

Complete addition laws for elliptic curves

D. J. Bernstein

University of Illinois at Chicago

Tanja Lange

Technische Universiteit Eindhoven

Weierstrass coordinates

Fix a field k with $2 \neq 0$.

Fix $a, b \in k$ with $4a^3 + 27b^2 \neq 0$.

Well-known fact:

The points of the “elliptic curve”
 $E : y^2 = x^3 + ax + b$ over k
form a commutative group $E(k)$.

“So the group is $\{(x, y) \in k \times k : y^2 = x^3 + ax + b\}$?”

Not exactly! It's $\{(x, y) \in k \times k : y^2 = x^3 + ax + b\} \cup \{\infty\}$.

To add $(x_1, y_1), (x_2, y_2) \in E(k)$:

Define $x_3 = \lambda^2 - x_1 - x_2$

and $y_3 = \lambda(x_1 - x_3) - y_1$

where $\lambda = (y_2 - y_1)/(x_2 - x_1)$.

Then $(x_3, y_3) \in E(k)$.

Geometric interpretation:

$(x_1, y_1), (x_2, y_2), (x_3, -y_3)$ are
on the curve $y^2 = x^3 + ax + b$
and on a line;

$(x_3, y_3), (x_3, -y_3)$ are
on a vertical line.

“So that’s the group law?

$(x_1, y_1) + (x_2, y_2) = (x_3, y_3)$?”

Not exactly! Definition of λ assumes that $x_2 \neq x_1$.

To add $(x_1, y_1), (x_1, y_1) \in E(k)$:

Define $x_3 = \lambda^2 - x_1 - x_2$

and $y_3 = \lambda(x_1 - x_3) - y_1$

where $\lambda = (3x_1^2 + a)/2y_1$.

Then $(x_3, y_3) \in E(k)$.

Geometric interpretation:

The curve's tangent line at

(x_1, y_1) passes through $(x_3, -y_3)$.

“So that's the group law?

One special case for doubling?”

Not exactly! More exceptions:
e.g., y_1 could be 0.

Six cases overall: $\infty + \infty = \infty$;

$$\infty + (x_2, y_2) = (x_2, y_2);$$

$$(x_1, y_1) + \infty = (x_1, y_1);$$

$$(x_1, y_1) + (x_1, -y_1) = \infty;$$

for $y_1 \neq 0$, $(x_1, y_1) + (x_1, y_1) = (x_3, y_3)$ with $x_3 = \lambda^2 - x_1 - x_2$,

$$y_3 = \lambda(x_1 - x_3) - y_1,$$

$$\lambda = (3x_1^2 + a)/2y_1;$$

for $x_1 \neq x_2$, $(x_1, y_1) + (x_2, y_2) = (x_3, y_3)$ with $x_3 = \lambda^2 - x_1 - x_2$,

$$y_3 = \lambda(x_1 - x_3) - y_1,$$

$$\lambda = (y_2 - y_1)/(x_2 - x_1).$$

$E(k)$ is a commutative group:

Has neutral element ∞ , and $-$:

$$-\infty = \infty; -(x, y) = (x, -y).$$

Commutativity: $P + Q = Q + P$.

Associativity:

$$(P + Q) + R = P + (Q + R).$$

Straightforward but tedious:

use a computer-algebra system
to check each possible case.

Or relate each $P + Q$ case
to “ideal-class product.”

Many other proofs,

but can't escape case analysis.

Projective coordinates

Can eliminate some exceptions.

Define $(X : Y : Z)$, for

$(X, Y, Z) \in k \times k \times k - \{(0, 0, 0)\}$,
as $\{(rX, rY, rZ) : r \in k - \{0\}\}$.

Could split into cases:

$$(X : Y : Z) =$$

$$(X/Z : Y/Z : 1) \text{ if } Z \neq 0;$$

$$(X : Y : 0) =$$

$$(X/Y : 1 : 0) \text{ if } Y \neq 0;$$

$$(X : 0 : 0) = (1 : 0 : 0).$$

But scaling unifies all cases.

Write $\mathbf{P}^2(k) = \{(X : Y : Z)\}$.

Revised definition: $E(k) = \{(X : Y : Z) \in \mathbf{P}^2(k) : Y^2 Z = X^3 + aXZ^2 + bZ^3\}$.

Could split into cases:

If $(X : Y : Z) \in E(k)$ and $Z \neq 0$:

$$(X : Y : Z) = (x : y : 1)$$

where $x = X/Z$, $y = Y/Z$.

Note that $y^2 = x^3 + ax + b$.

Corresponds to previous (x, y) .

If $(X : Y : Z) \in E(k)$ and $Z = 0$:

$$X^3 = 0 \text{ so } X = 0 \text{ so } Y \neq 0$$

$$\text{so } (X : Y : Z) = (0 : 1 : 0).$$

Corresponds to previous ∞ .

$$\begin{aligned}
& (X_1 : Y_1 : Z_1) + (X_2 : Y_2 : Z_2) \\
&= (X_3 : Y_3 : Z_3) \text{ where} \\
& U = Y_2 Z_1 - Y_1 Z_2, \\
& V = X_2 Z_1 - X_1 Z_2, \\
& W = U^2 Z_1 Z_2 - V^3 - 2V^2 X_1 Z_2, \\
& X_3 = VW, \\
& Y_3 = U(V^2 X_1 Z_2 - W) - V^3 Y_1 Z_2, \\
& Z_3 = V^3 Z_1 Z_2.
\end{aligned}$$

“Aha! No more divisions by 0.”

Compare to previous formulas:

$$x_3 = \lambda^2 - x_1 - x_2$$

$$\text{and } y_3 = \lambda(x_1 - x_3) - y_1$$

$$\text{where } \lambda = (y_2 - y_1)/(x_2 - x_1).$$

Oops, still have exceptions!

Formulas give bogus

$$(X_3, Y_3, Z_3) = (0, 0, 0)$$

if $(X_1 : Y_1 : Z_1) = (0 : 1 : 0)$.

Same problem for doubling.

Formulas produce $(0 : 1 : 0)$ for

$$(X_1 : Y_1 : Z_1) + (X_1 : -Y_1 : Z_1)$$

if $Y_1 \neq 0$ and $Z_1 \neq 0$

but not if $Y_1 = 0$.

To define complete group law,
use six cases as before.

Jacobian coordinates

“Weighted projective coordinates using weights 2, 3, 1”:

Redefine $(X : Y : Z)$ as $\{(r^2 X, r^3 Y, r Z) : r \in k - \{0\}\}$.

Redefine $E(k)$ using $Y^2 = X^3 + aXZ^4 + bZ^6$.

Could again split into cases for $(X : Y : Z) \in E(k)$:
if $Z \neq 0$ then $(X : Y : Z) = (X/Z^2 : Y/Z^3 : 1)$; if $Z = 0$ then $(X : Y : Z) = (1 : 1 : 0)$.

$$\begin{aligned}
& (X_1 : Y_1 : Z_1) + (X_2 : Y_2 : Z_2) \\
&= (X_3 : Y_3 : Z_3) \text{ where} \\
& U_1 = X_1 Z_2^2, \quad U_2 = X_2 Z_1^2, \\
& S_1 = Y_1 Z_2^3, \quad S_2 = Y_2 Z_1^3, \\
& H = U_2 - U_1, \quad J = S_2 - S_1, \\
& X_3 = -H^3 - 2U_1 H^2 + J^2, \\
& Y_3 = -S_1 H^3 + J(U_1 H^2 - X_3), \\
& Z_3 = Z_1 Z_2 H.
\end{aligned}$$

Streamlined algorithm

uses $12\mathbf{M} + 4\mathbf{S}$, where

$\mathbf{S}$ is squaring in k and

$\mathbf{M}$ is general multiplication in k .

(1986 Chudnovsky–Chudnovsky)

$11\mathbf{M} + 5\mathbf{S}$. (2001 Bernstein)

Still need all six cases.

Why use Jacobian coordinates?

Answer: Only $3\mathbf{M} + 5\mathbf{S}$

for Jacobian-coordinate doubling
if $a = -3$ (e.g. NIST curves).

Formulas: If $Y_1 \neq 0$ then

$$(X_1 : Y_1 : Z_1) + (X_1 : Y_1 : Z_1)$$

$$= (X_3, Y_3, Z_3) \text{ where}$$

$$T = Z_1^2, U = Y_1^2, V = X_1 U,$$

$$W = 3(X_1 - T)(X_1 + T),$$

$$X_3 = W^2 - 8V,$$

$$Z_3 = (Y_1 + Z_1)^2 - U - T,$$

$$Y_3 = W(4V - X_3) - 8U^2.$$

Unified addition laws

Do addition laws
have to fail for doublings?
Not necessarily!

Example: “Jacobi intersection”

$$s^2 + c^2 = 1, as^2 + d^2 = 1$$

has **17M** addition formula
that works for doublings.

(1986 Chudnovsky–Chudnovsky)

16M. (2001 Liardet–Smart)

Many more “unified formulas.”

But always find exceptions:
points not added by formulas.

“Is this Jacobi intersection
related to $y^2 = x^3 + \dots$?”

Yes: $s^2 + c^2 = 1$, $as^2 + d^2 = 1$

is birationally equivalent to

$$y^2 = x^3 + (2 - a)x^2 + (1 - a)x.$$

$$(s, c, d) \mapsto (x, y):$$

$$x = (d - 1)(1 - a)/(ca - d + 1 - a);$$

$$y = s(1 - a)a/(ca - d + 1 - a).$$

$$(x, y) \mapsto (s, c, d):$$

$$s = -2y/((y^2/x^2 + a)x);$$

$$c = 1 - 2/(y^2/x^2 + a) - \\ 2(1 - a)/((y^2/x^2 + a)x);$$

$$d = 1 - 2a/(y^2/x^2 + a).$$

Do we need 6 cases? No!

Can cover $E(k) \times E(k)$

using 3 addition laws.

(1985 H. Lange–Ruppert)

How about just *one* law

that covers $E(k) \times E(k)$?

One complete addition law?

Bad news: “Theorem 1.

The smallest cardinality of a complete system of addition laws on E equals two.”

(1995 Bosma–H. Lenstra)

Edwards curves

2007 Edwards:

Every elliptic curve over $\overline{\mathbf{Q}}$ is birationally equivalent to $x^2 + y^2 = c^2(1 + x^2y^2)$ for some $c \in \overline{\mathbf{Q}} - \{0, \pm 1, \pm i\}$.

$x^2 + y^2 = c^2(1 + x^2y^2)$ has neutral element $(0, c)$, addition $(x_1, y_1) + (x_2, y_2) = (x_3, y_3)$ with

$$x_3 = \frac{x_1y_2 + y_1x_2}{c(1 + x_1x_2y_1y_2)},$$

$$y_3 = \frac{y_1y_2 - x_1x_2}{c(1 - x_1x_2y_1y_2)}.$$

2007 Bernstein–Lange:

Over a non-binary finite field k ,

$$x^2 + y^2 = c^2(1 + dx^2y^2)$$

covers more elliptic curves.

Here $c, d \in k^*$ with $dc^4 \neq 1$.

$$x_3 = \frac{x_1y_2 + y_1x_2}{c(1 + dx_1x_2y_1y_2)},$$

$$y_3 = \frac{y_1y_2 - x_1x_2}{c(1 - dx_1x_2y_1y_2)}.$$

Can always take $c = 1$. Then

$10\mathbf{M} + 1\mathbf{S} + 1\mathbf{D}$ for addition,

$3\mathbf{M} + 4\mathbf{S}$ for doubling.

Latest news, comparisons:

hyperelliptic.org/EFD

Completeness

2007 Bernstein–Lange:

If d is not a square in k then

$$\{(x, y) \in k \times k : \\ x^2 + y^2 = c^2(1 + dx^2y^2)\}$$

is a commutative group
under this addition law.

The denominators

$$c(1 + dx_1x_2y_1y_2),$$

$$c(1 - dx_1x_2y_1y_2)$$

are never zero.

No exceptional cases!

Recall Bosma–Lenstra theorem:

“The smallest cardinality of a complete system of addition laws on E equals two.”

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“The smallest cardinality of a complete system of addition laws on E equals two.” . . . meaning:

Any addition formula

for a Weierstrass curve E

in projective coordinates

must have exceptional cases

in $E(\bar{k}) \times E(\bar{k})$, where

$\bar{k}$ = algebraic closure of k .

Recall Bosma–Lenstra theorem:

“The smallest cardinality of a complete system of addition laws on E equals two.” . . . meaning:

Any addition formula

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in $E(\bar{k}) \times E(\bar{k})$, where

$\bar{k}$ = algebraic closure of k .

Edwards addition formula has

exceptional cases for $E(\bar{k})$

. . . but not for $E(k)$.

We do computations in $E(k)$.

Cryptographic impact

Advantages for cryptography
of choosing Edwards curves:

Very high speed.

Completeness eases
implementations, avoids
simple side-channel attacks.

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Oops, hardware people
want binary fields!

2008 B.–L.–Rezaeian Farashahi:
binary analogue to Edwards
curves; complete, very fast.

Still one reason for complaint.

Edwards curves always have point of order 4.

Standard NIST curves were chosen to have prime order.

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NIST curves can't take advantage of Edwards speed *and* don't have complete addition formulas.

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NIST curves can't take advantage of Edwards speed *and* don't have complete addition formulas.

2009 Bernstein–Lange, this talk:
Have a complete addition law for all of these curves.

Today's curve shape

Fix a field k with $2 \neq 0$.

Fix $t, d \in k$ with $d \neq 0$,
 $d \neq (t + 2)^2$, $27d \neq (2 - t)^3$.

Consider the curve

$$x^2 + y^2 = x + y + txy + dx^2y^2$$

with neutral element $(0, 0)$.

Warning: We're still studying
choices of curve shapes; we don't
promise that this is the best.

For comparison, Edwards:

$$x^2 + y^2 = 1 + dx^2y^2$$

with neutral element $(0, 1)$.

Birational equivalence from

$$x^2 + y^2 = x + y + txy + dx^2y^2 \text{ to}$$

$$v^2 - (t + 2)uv + dv =$$

$$u^3 - (t + 2)u^2 - du + (t + 2)d$$

$$\text{i.e. } v^2 - (t + 2)uv + dv =$$

$$(u^2 - d)(u - (t + 2)):$$

$$u = (dxy + t + 2)/(x + y);$$

$$v = \frac{((t + 2)^2 - d)x}{(t + 2)xy + x + y}.$$

Assuming $t + 2$ square, d not:

only exceptional point is

$(0, 0)$, mapping to ∞ .

$$\text{Inverse: } x = v/(u^2 - d);$$

$$y = ((t + 2)u - v - d)/(u^2 - d).$$

Example: the NIST curves

Consider curve with $d = -1$ and

$$t = \begin{array}{l} 77856058252666544098227759201 \\ 8607150561834371823249249461 \end{array}$$

over $\mathbf{F}_p$ where $p = 2^{192} - 2^{64} - 1$.

Note: d is non-square in $\mathbf{F}_p$.

Birationally equivalent to
standard “NIST P-192” curve

$$v^2 = u^3 - 3u + a_6 \text{ where}$$

$$a_6 = \begin{array}{l} 24551555460089438177402939151 \\ 97451784769108058161191238065 \end{array}$$

Consider curve with $d = 11$ and

$$t = \begin{array}{r} 89561265817923268463529369784 \\ 59653337798320066750209233023 \\ 6009670 \end{array}$$

over $\mathbf{F}_p$ where $p = 2^{224} - 2^{96} + 1$.

Note: d is non-square in $\mathbf{F}_p$.

Birationally equivalent to
standard “NIST P-224” curve

$$v^2 = u^3 - 3u + a_6 \text{ where}$$

$$a_6 = \begin{array}{r} 18958286285566608000408668544 \\ 49392641550468096867932107578. \\ 7234672564 \end{array}$$

Consider curve with $d = -1$ and

$$t = \begin{array}{r} 78751018041117252545420999954 \\ 76717646453854506081463020284 \\ 1395651175859201799 \end{array}$$

over $\mathbf{F}_p$ where $p = 2^{256} - 2^{224} + 2^{192} + 2^{96} - 1$.

Note: d is non-square in $\mathbf{F}_p$.

Birationally equivalent to
standard “NIST P-256” curve
 $v^2 = u^3 - 3u + a_6$ where

$$a_6 = \begin{array}{r} 41058363725152142129326129780 \\ 04726840911444101599372555483. \\ 5256314039467401291 \end{array}$$

Consider curve with $d = -1$ and

$$t = \begin{array}{r} 85909296364310935634030366769 \\ 37570960716721909626687223623 \\ 19596768294026516624086336448 \\ 0501907705272975221538249252 \end{array}$$

over $\mathbf{F}_p$ where $p = 2^{384} - 2^{128} - 2^{96} + 2^{32} - 1$.

Note: d is non-square in $\mathbf{F}_p$.

Birationally equivalent to
standard “NIST P-384” curve

$v^2 = u^3 - 3u + a_6$ where

$$a_6 = \begin{array}{r} 75801935599597058778490118403 \\ 89048093056905856361568521428 \\ 70730198868924130986086513626 \\ 0764883745107765439761230575 \end{array}$$

Consider curve with $d = 3$ and

$$t = \begin{array}{r} 28255491549159851139291566929 \\ 14423222253417506441326327182 \\ 78098467340130883832560776891 \\ 27881593298389934213527989123 \\ 13871892632272472360900308353 \\ 04279675250 \end{array}$$

over $\mathbf{F}_p$ where $p = 2^{521} - 1$.

Note: d is non-square in $\mathbf{F}_p$.

Birationally equivalent to
standard “NIST P-521” curve

$v^2 = u^3 - 3u + a_6$ where

$$a_6 = \begin{array}{r} 10938490380737342745111123907 \\ 66805569936207598951683748994 \\ 58639449595311615073501601370 \\ 87375737596232485921322967063 \\ 13309438452531591012912142327 \\ 488478985984 \end{array}$$

Today's addition law

$$x_3 = \frac{x_1 + x_2 + (t - 2)x_1x_2 + (x_1 - y_1)(x_2 - y_2) + dx_1^2(x_2y_1 + x_2y_2 - y_1y_2)}{1 - 2dx_1x_2y_2 - dx_1^2(x_2 + y_2 + (t - 2)x_2y_2)};$$

$$y_3 = \frac{y_1 + y_2 + (t - 2)y_1y_2 + (y_1 - x_1)(y_2 - x_2) + dy_1^2(y_2x_1 + y_2x_2 - x_1x_2)}{1 - 2dy_1y_2x_2 - dy_1^2(y_2 + x_2 + (t - 2)y_2x_2)}.$$

Exercise: On curve,
if denominators are nonzero.

Exercise: $(x, y) + (0, 0) = (x, y)$.

Exercise: $(x, y) + (y, x) = (0, 0)$.

Exercise: Compute projectively
using $26\mathbf{M} + 8\mathbf{S} + 8\mathbf{D}$.

... Clearly can be improved;
we're not done optimizing yet.

Exercise: Corresponds to
addition on Weierstrass curve.

Completeness

$$x_3 = \frac{x_1 + x_2 + (t - 2)x_1x_2 + (x_1 - y_1)(x_2 - y_2) + dx_1^2(x_2y_1 + x_2y_2 - y_1y_2)}{1 - 2dx_1x_2y_2 - dx_1^2(x_2 + y_2 + (t - 2)x_2y_2)};$$

$$y_3 = \frac{y_1 + y_2 + (t - 2)y_1y_2 + (y_1 - x_1)(y_2 - x_2) + dy_1^2(y_2x_1 + y_2x_2 - x_1x_2)}{1 - 2dy_1y_2x_2 - dy_1^2(y_2 + x_2 + (t - 2)y_2x_2)}.$$

Can denominators be 0?

Only if d is a square!

Theorem: Assume that

k is a field with $2 \neq 0$;

$d, t, x_1, y_1, x_2, y_2 \in k$;

d is not a square in k ;

$27d \neq (2 - t)^3$;

$x_1^2 + y_1^2 = x_1 + y_1 + tx_1y_1 + dx_1^2y_1^2$;

$x_2^2 + y_2^2 = x_2 + y_2 + tx_2y_2 + dx_2^2y_2^2$.

Then $1 - 2dx_1x_2y_2 -$

$dx_1^2(x_2 + y_2 + (t - 2)x_2y_2) \neq 0$.

Only if d is a square!

Theorem: Assume that

k is a field with $2 \neq 0$;

$d, t, x_1, y_1, x_2, y_2 \in k$;

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$x_1^2 + y_1^2 = x_1 + y_1 + tx_1y_1 + dx_1^2y_1^2$;

$x_2^2 + y_2^2 = x_2 + y_2 + tx_2y_2 + dx_2^2y_2^2$.

Then $1 - 2dx_1x_2y_2 -$

$dx_1^2(x_2 + y_2 + (t - 2)x_2y_2) \neq 0$.

By $x \leftrightarrow y$ symmetry

also $1 - 2dy_1y_2x_2 -$

$dy_1^2(y_2 + x_2 + (t - 2)y_2x_2) \neq 0$.

Proof: Suppose that

$$1 - 2dx_1x_2y_2 -$$

$$dx_1^2(x_2 + y_2 + (t - 2)x_2y_2) = 0.$$

Proof: Suppose that

$$1 - 2dx_1x_2y_2 - dx_1^2(x_2 + y_2 + (t - 2)x_2y_2) = 0.$$

Note that $x_1 \neq 0$.

Proof: Suppose that

$$1 - 2dx_1x_2y_2 - dx_1^2(x_2 + y_2 + (t - 2)x_2y_2) = 0.$$

Note that $x_1 \neq 0$.

Use curve equation₂ to see that

$$(1 - dx_1x_2y_2)^2 = dx_1^2(x_2 - y_2)^2.$$

Proof: Suppose that

$$1 - 2dx_1x_2y_2 - dx_1^2(x_2 + y_2 + (t - 2)x_2y_2) = 0.$$

Note that $x_1 \neq 0$.

Use curve equation₂ to see that

$$(1 - dx_1x_2y_2)^2 = dx_1^2(x_2 - y_2)^2.$$

By hypothesis d is non-square

$$\text{so } x_1^2(x_2 - y_2)^2 = 0$$

$$\text{and } (1 - dx_1x_2y_2)^2 = 0.$$

Proof: Suppose that

$$1 - 2dx_1x_2y_2 -$$

$$dx_1^2(x_2 + y_2 + (t - 2)x_2y_2) = 0.$$

Note that $x_1 \neq 0$.

Use curve equation₂ to see that

$$(1 - dx_1x_2y_2)^2 = dx_1^2(x_2 - y_2)^2.$$

By hypothesis d is non-square

$$\text{so } x_1^2(x_2 - y_2)^2 = 0$$

$$\text{and } (1 - dx_1x_2y_2)^2 = 0.$$

Hence $x_2 = y_2$ and $1 = dx_1x_2y_2$.

Curve equation₁ times $1/x_1^2$:

$$1 + y_1^2/x_1^2 =$$

$$1/x_1 + y_1(1/x_1^2 + t/x_1) + dy_1^2.$$

Curve equation₁ times $1/x_1^2$:

$$1 + y_1^2/x_1^2 =$$

$$1/x_1 + y_1(1/x_1^2 + t/x_1) + dy_1^2.$$

Substitute $1/x_1 = dx_2^2$:

$$1 + d^2 y_1^2 x_2^4 =$$

$$dx_2^2 + dy_1(dx_2^4 + x_2^2 t) + dy_1^2.$$

Curve equation₁ times $1/x_1^2$:

$$1 + y_1^2/x_1^2 =$$

$$1/x_1 + y_1(1/x_1^2 + t/x_1) + dy_1^2.$$

Substitute $1/x_1 = dx_2^2$:

$$1 + d^2 y_1^2 x_2^4 =$$

$$dx_2^2 + dy_1(dx_2^4 + x_2^2 t) + dy_1^2.$$

Substitute $2x_2^2 = 2x_2 + tx_2^2 + dx_2^4$:

$$(1 - dy_1 x_2^2)^2 = d(x_2 - y_1)^2.$$

Curve equation₁ times $1/x_1^2$:

$$1 + y_1^2/x_1^2 =$$

$$1/x_1 + y_1(1/x_1^2 + t/x_1) + dy_1^2.$$

Substitute $1/x_1 = dx_2^2$:

$$1 + d^2 y_1^2 x_2^4 =$$

$$dx_2^2 + dy_1(dx_2^4 + x_2^2 t) + dy_1^2.$$

Substitute $2x_2^2 = 2x_2 + tx_2^2 + dx_2^4$:

$$(1 - dy_1 x_2^2)^2 = d(x_2 - y_1)^2.$$

Thus $x_2 = y_1$ and $1 = dy_1 x_2^2$.

Hence $1 = dx_2^3$.

Curve equation₁ times $1/x_1^2$:

$$1 + y_1^2/x_1^2 =$$

$$1/x_1 + y_1(1/x_1^2 + t/x_1) + dy_1^2.$$

Substitute $1/x_1 = dx_2^2$:

$$1 + d^2 y_1^2 x_2^4 =$$

$$dx_2^2 + dy_1(dx_2^4 + x_2^2 t) + dy_1^2.$$

Substitute $2x_2^2 = 2x_2 + tx_2^2 + dx_2^4$:

$$(1 - dy_1 x_2^2)^2 = d(x_2 - y_1)^2.$$

Thus $x_2 = y_1$ and $1 = dy_1 x_2^2$.

Hence $1 = dx_2^3$.

Now $2x_2^2 = 2x_2 + tx_2^2 + x_2$

so $3 = (2-t)x_2$ so $27d = (2-t)^3$.

Contradiction.

