Breaking ECC2K-130

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2009.12.04

Prehistoric Diffie-Hellman



Common Key: $k = g^{ab}$ can be used in symmetric crypto. **Discrete-Logarithm Problem** (DLP): Given g, h_A find a.

Index calculus

"Index-calculus attacks" break the DLP in F_{p}^{*} , or more generally F_{q}^{*} , in subexponential time:

- Classic index calculus: time 2^{O((|g n)^{1/2}(|g |g n)^{1/2})} for n-bit primes.
- Number-field sieve (NFS): time 2^{O((|g n)^{1/3}(|g |g n)^{2/3})} for *n*-bit primes.

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1985 Miller, 1987 Koblitz (before NFS!): Replace \mathbf{F}_q^* with an elliptic curve over \mathbf{F}_q . Index-calculus attacks will fail, so can use much smaller q.

Subsequent analysis has found a few elliptic curves vulnerable to index calculus, but almost all curves seem safe.

Elliptic curves

Long Weierstrass form for an elliptic curve over a field k:

$$y^{2} + \underbrace{(a_{1}x + a_{3})}_{h(x)} y = \underbrace{x^{3} + a_{2}x^{2} + a_{4}x + a_{6}}_{f(x)}, h, f \in k[x].$$

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Can take almost any $a_1, a_2, a_3, a_4, a_6 \in k$. Just one requirement: the curve is nonsingular; i.e., no point $(x_1, y_1) \in \overline{k} \times \overline{k}$ satisfies simultaneously $y_1^2 + h(x_1)y_1 = f(x_1), 2y_1 + h(x_1) = 0, h'(x_1)y_1 = f'(x_1)$.

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For an overview of other (often faster!) coordinate systems, see the EFD: http://hyperelliptic.org/EFD/

One popular choice: $k = F_{2^n}$. Fast hardware implementations. Some index-calculus threats for composite *n* and for n = 127, but the bad cases are easy to recognize and avoid.

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For odd n, each curve can be transformed to one of two forms:

$$y^2 + y = x^3 + a_4 x + a_6, \ a_4, a_6 \in \mathsf{F}_{2^n};$$

 $y^2 + xy = x^3 + a_2 x^2 + a_6, \ a_2, a_6 \in \mathsf{F}_{2^n}.$

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Curves of the second form are "ordinary". Transfers are known for, e.g., order $2(2^{2p} - 2^p + 1)$, but almost all curves seem safe.

Arithmetic on ordinary binary curves

Fix $E: y^2 + xy = x^3 + a_2x^2 + a_6$ with $a_2, a_6 \in F_{2^n}$. Elements of the group $E(F_{2^n})$: a special point P_{∞} , and each $(x_1, y_1) \in F_{2^n} \times F_{2^n}$ satisfying $y_1^2 + x_1y_1 = x_1^3 + a_2x_1^2 + a_6$.

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How to add $P_1, P_2 \in E(\mathbf{F}_{2^n})$:

• $P_1 + P_{\infty} = P_{\infty} + P_1 = P_1$; i.e., P_{∞} is neutral.

•
$$(x_1, y_1) + (x_1, y_1 + x_1) = P_{\infty}$$
.

• If $x_1 \neq 0$ the double $[2](x_1, y_1) = (x_3, y_3)$ is given by

$$x_3 = \lambda^2 + \lambda + a_2, \ y_3 = \lambda(x_1 + x_3) + y_1 + x_3, \ \text{where} \ \lambda = x_1 + y_1/x_1.$$

• If
$$x_1
eq x_2$$
 the sum $(x_1, y_1) + (x_2, y_2) = (x_3, y_3)$ is given by

$$x_3 = \lambda^2 + \lambda + a_2 + x_1 + x_2, \ y_3 = \lambda(x_1 + x_3) + y_1 + x_3, \ ext{where} \ \lambda = rac{y_1 + y_2}{x_1 + x_2}$$

Cost: 11 (inversion), 2M (multiplications), 1S (squaring).

Koblitz curves

1991 Koblitz: Scalar multiplication $m, P \mapsto [m]P$ can be computed more efficiently on curves

$$E_a: y^2 + xy = x^3 + ax^2 + 1,$$
 where $a \in \{0, 1\}.$

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Main tool: the "Frobenius endomorphism" $\sigma: E_a(\mathbf{F}_{2^n}) \to E_a(\mathbf{F}_{2^n})$ defined by $\sigma((x_1, y_1)) = (x_1^2, y_1^2)$ and $\sigma(P_{\infty}) = P_{\infty}$.

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"The characteristic polynomial of Frobenius": $\sigma^2(P) + [\mu]\sigma(P) + [2]P = P_{\infty}$, where $\mu = 1$ for a = 0 and $\mu = -1$ for a = 1. Use this equation to replace a double-and-add sequence with a faster σ -and-add sequence.

Choosing ECC key sizes

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I can easily imagine an attacker with the resources to break a 160-bit elliptic curve in under a year. Users should not expose themselves to this risk; they should instead move up to the comfortable security level of Curve25519.

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2007 Oliveira-Aranha-Morais-Daguano-López-Dahab:

Until now, the [largest] sizes for which the ECDLP and the DLP in prime fields are known to be solved are 2^{109} [17] and 2^{448} [4], respectively. Therefore, it seems that $\ell \ge 2^{128}$ and $q^k \ge 2^{512}$ are able to meet the current security requirements of WSNs. 2009 Bos-Kaihara-Kleinjung-Lenstra-Montgomery:

We address a short term but nevertheless important question that many practitioners will face the next few years, namely until when the current standards, 1024-bit RSA and 160-bit ECC, can responsibly be used ...

In a decade, very optimistically incorporating 10-fold cryptanalytic advances, still millions of devices would be required, and a successful open community attack on 160-bit ECC even by the year 2020 must be considered very unlikely. ...

There does not seem to be any reason to be concerned about continued usage of 160-bit prime field ECC during the next decade.

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The Certicom challenges

1997: Certicom announces several ECDLP prizes:

The Challenge is to compute the ECC private keys from the given list of ECC public keys and associated system parameters. This is the type of problem facing an adversary who wishes to completely defeat an elliptic curve cryptosystem.

Objectives:

1. To increase the cryptographic community's understanding and appreciation of the difficulty of the ECDLP.

2. To confirm comparisons of the security levels of systems such as ECC, RSA and DSA that have been made based primarily on theoretical considerations.

3. To provide information on how users of elliptic curve public-key cryptosystems should select suitable key lengths for a desired level of security.

4. To determine whether there is any significant difference in the difficulty of the ECDLP for elliptic curves over F_{2^m} and the ECDLP for elliptic curves over F_p .

5. To determine whether there is any significant difference in the difficulty of the ECDLP for random elliptic curves over \mathbf{F}_{2^m} and the ECDLP for Koblitz curves.

6. To encourage and stimulate research in computational and algorithmic number theory and, in particular, the study of the ECDLP.

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The Certicom challenges, level 0: exercises

		"Estimated number	
Bits	Name	of machine days"	Prize
79	ECCp-79	146	book
79	ECC2-79	352	book
89	ECCp-89	4360	book
89	ECC2-89	11278	book
97	ECC2K-95	8637	\$5000
97	ECCp-97	71982	\$5000
97	ECC2-97	180448	\$5000

Certicom believes that it is feasible that the 79-bit exercises could be solved in a matter of hours, the 89-bit exercises could be solved in a matter of days, and the 97-bit exercises in a matter of weeks using a network of 3000 computers.

The Certicom challenges, level 1

		"Estimated number	
Bits	Name	of machine days"	Prize
109	ECC2K-108	1300000	\$10000
109	ECCp-109	900000	\$10000
109	ECC2-109	2100000	\$10000
131	ECC2K-130	270000000	\$20000
131	ECCp-131	2300000000	\$20000
131	ECC2-131	66000000000	\$20000

The 109-bit Level I challenges are feasible using a very large network of computers. The 131-bit Level I challenges are expected to be infeasible against realistic software and hardware attacks, unless of course, a new algorithm for the ECDLP is discovered.

The Certicom challenges, level 2

		"Estimated number	
Bits	Name	of machine days"	Prize
163	ECC2K-163	32000000000000	\$30000
163	ECCp-163	230000000000000000000000000000000000000	\$30000
163	ECC2-163	620000000000000000000000000000000000000	\$30000
191	ECCp-191	48000000000000000000000	\$40000
191	ECC2-191	100000000000000000000000000000000000000	\$40000
239	ECC2K-238	920000000000000000000000000000000000000	\$50000
239	ECCp-239	140000000000000000000000000000000000000	\$50000
239	ECC2-238	210000000000000000000000000000000000000	\$50000
359	ECCp-359	$\approx \infty$	\$100000

The Level II challenges are infeasible given today's computer technology and knowledge.

Broken challenges

- 1997: Baisley and Harley break ECCp-79.
 1997: Harley et al. break ECC2-79.
 1998: Harley et al. break ECCp-89.
 1998: Harley et al. break ECC2-89.
 1998: Harley et al. (1288 computers) break ECCp-97.
 1998: Harley et al. (200 computers) break ECC2K-95.
 1999: Harley et al. (740 computers) break ECC2-97.
- 2000: Harley et al. (9500 computers) break ECC2K-108.

Updated cert_ecc_challenge.pdf still says "109-bit Level I challenges are feasible using a very large network ... 131-bit Level I challenges are expected to be infeasible" etc.

2002: Monico et al. (10000 computers) break ECCp-109. 2004: Monico et al. (2600 computers) break ECC2-109.

The target: ECC2K-130

The Koblitz curve $y^2 + xy = x^3 + 1$ over $F_{2^{131}} = F_2[z]/(z^{131} + z^{13} + z^2 + z + 1)$ has 4ℓ points, where ℓ is the prime $680564733841876926932320129493409985129 \approx 2^{129}$.

Certicom generated two random points on the curve and multiplied them by 4, obtaining the following points P, Q:

x(P) = 05 1C99BFA6 F18DE467 C80C23B9 8C7994AA y(P) = 04 2EA2D112 ECEC71FC F7E000D7 EFC978BD x(Q) = 06 C997F3E7 F2C66A4A 5D2FDA13 756A37B1y(Q) = 04 A38D1182 9D32D347 BD0C0F58 4D546E9A

The challenge:

Find an integer $k \in \{0, 1, \dots, \ell - 1\}$ such that [k]P = Q.

The attacker: ECRYPT

European Union has funded ECRYPT | network (2004–2008) and now ECRYPT || network (2008–2012).

ECRYPT II has 11 partners (KU Leuven, ENS, EPFL, RU Bochum, RHUL, TU Eindhoven, TU Graz, U Bristol, U Salerno, France Télécom, IBM Research), 22 adjoint members.

ECRYPT II work is handled by three "virtual labs":

- SymLab: "Symmetric Techniques";
- MAYA: "Multi-party and asymmetric algorithms";
- ► VAMPIRE: "Applications and Implementations".

Working groups in VAMPIRE:

- ► VAM1: "Efficient Implementation of Security Systems".
- VAM2: "Physical Security".

ECRYPT vs. ECC2K-130

2009.02: VAMPIRE VAM1 sets its sights on ECC2K-130. Optimizing ECC attacks isn't far from optimizing ECC. Exactly how difficult is breaking ECC2K-130?

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2009.12: With our latest implementations, ECC2K-130 is breakable in a year on average

- ▶ by 3039 3GHz Core 2 CPUs,
- ▶ or by 2716 GTX 295 GPUs,
- ▶ or by 2466 Cell CPUs,
- or by 2026 XC3S5000 FPGAs,
- ▶ or by (estimated) 200 ASICs costing 60000 EUR,
- or by any combination thereof.

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- or by any combination thereof.
- This is what Certicom called "infeasible"?

The most important ECDL algorithms

No known index-calculus attack applies to ECC2K-130. But can still use generic attacks that work in any group:

- ► The Pohlig-Hellman attack reduces the hardness of the ECDLP to the hardness of the ECDLP in the largest subgroup of prime order: in this case order *l*.
- ▶ The Baby-Step Giant-Step attack finds the logarithm in $\sqrt{\ell}$ steps and $\sqrt{\ell}$ storage by comparing Q [jt]P (the giant steps) to a sorted list of all [i]P (the baby steps), where $0 \le i, j \le \lceil \sqrt{\ell} \rceil$ and $t = \lceil \sqrt{\ell} \rceil$.
- ► Pollard's rho and kangaroo methods also use O(√ℓ) steps but require constant memory—much less expensive! The kangaroo method would be faster if the logarithm were known to lie in a short interval; for us rho is best.
- Multiple-target attacks: not relevant here.

Pollard's rho method

Make a pseudo-random walk in $\langle P \rangle$, where the next step depends on current point: $P_{i+1} = f(P_i)$.

Birthday paradox: Randomly choosing from ℓ elements picks one element twice after about $\sqrt{\pi\ell/2}$ draws.

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Assume that for each point we know $a_i, b_i \in \mathbb{Z}/\ell\mathbb{Z}$ so that $P_i = [a_i]P + [b_i]Q$. Then $P_i = P_j$ means that

 $[a_i]P + [b_i]Q = [a_j]P + [b_j]Q$ so $[b_i - b_j]Q = [a_j - a_i]P$.

If $b_i \neq b_j$ the ECDLP is solved: $k = (a_j - a_i)/(b_i - b_j)$.

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 so $[b_i - b_j]Q = [a_j - a_i]P$.

If $b_i \neq b_j$ the ECDLP is solved: $k = (a_j - a_i)/(b_i - b_j)$. e.g. "Adding walk": Start with $P_0 = P$ and put $f(P_i) = P_i + [c_r]P + [d_r]Q$ where $r = h(P_i)$.
A rho within a random walk on 1024 elements



Method is called rho method because of the shape.

Parallel collision search

Running Pollard's rho method on N computers gives speedup of $\approx \sqrt{N}$ from increased likelihood of finding collision.

Want better way to spread computation across clients. Want to find collisions between walks on different machines, without frequent synchronization!

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Want better way to spread computation across clients. Want to find collisions between walks on different machines, without frequent synchronization!

Perform walks with different starting points but same update function on all computers. If same point is found on two different computers also the following steps will be the same.

Terminate each walk once it hits a distinguished point. Attacker chooses definition of distinguished points; can be more or less frequent. Do not wait for cycle.

Collect all distinguished points in central database.

Expect collision within $O(\sqrt{\ell}/N)$ iterations. Speedup $\approx N$.

Short walks ending in distinguished points



Blue and orange paths found the same distinguished point!

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Equivalence classes

P and -P have same *x*-coordinate. Search for *x*-coordinate collision. Search space for collisions is only $\ell/2$; this gives factor $\sqrt{2}$ speedup ... provided that $f(P_i) = f(-P_i)$.

Solution: $f(P_i) = |P_i| + [c_r]P + [d_r]Q$ where $r = h(|P_i|)$. Define $|P_i|$ as, e.g., lexicographic minimum of $P_i, -P_i$.

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Problem: this walk can run into fruitless cycles! If there are S different steps $[c_r]P + [d_r]Q$ then with probability $1/(4S^2)$ the following happens:

$$P_{i+2} = -P_{i+1} + [c_r]P + [d_r]Q$$

= -(-P_i + [c_r]P + [d_r]Q) + [c_r]P + [d_r]Q = P_i.

Get $P_{i+3} = P_{i+1}$, $P_{i+4} = P_i$, etc. Can detect and fix. Some effort; not exactly $\sqrt{2}$ speedup.

Equivalence classes for Koblitz curves

More savings: P and $\sigma^i(P)$ have $x(\sigma^j(P)) = x(P)^{2^j}$.

Reduce number of iterations by another factor \sqrt{n} by considering equivalence classes under Frobenius and \pm .

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Need to ensure that the iteration function satisfies $f(P_i) = f(\pm \sigma^j(P_i))$.

Equivalence classes for Koblitz curves

More savings: P and $\sigma^i(P)$ have $x(\sigma^j(P)) = x(P)^{2^j}$.

Reduce number of iterations by another factor \sqrt{n} by considering equivalence classes under Frobenius and \pm .

Need to ensure that the iteration function satisfies $f(P_i) = f(\pm \sigma^j(P_i))$.

Could again define adding walk starting from $|P_i|$. Redefine $|P_i|$ as canonical representative of class containing P_i : e.g., lexicographic minimum of P_i , $-P_i$, $\sigma(P_i)$, etc.

Iterations now involve many squarings, but squarings are not so expensive in characteristic 2.

Our choice of iteration function

In normal basis, x(P) and $x(P)^{2^{j}}$ have same Hamming weight HW(x(P)). Convenient to use this to determine iteration.

Our iteration function—note that HW(x(P)) is always even:

$$P_{i+1} = P_i + \sigma^j (P_i),$$

where $j = (HW(x(P))/2 \mod 8) + 3$.

This nicely avoids short, fruitless cycles.

Iteration consists of

- computing the Hamming weight HW(x(P)) of the normal-basis representation of x(P);
- checking for distinguished points (is $HW(x(P)) \le 34$?);
- computing j and $P + \sigma^j(P)$.

Analysis of our choice of iteration function

For a perfectly random walk $\sqrt{\pi \ell/2}$ iterations are expected on average. Have $\ell \approx 2^{131}/4$ for ECC2K-130.

A perfectly random walk on classes under \pm and Frobenius would reduce number of iterations by $\sqrt{2\cdot 131}.$

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Loss of randomness from having only 8 choices of j. Further loss from non-randomness of Hamming weights: Hamming weights around 66 are much more likely than at the edges; effect still noticeable after reduction to 8 choices.

Analysis of our choice of iteration function

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Loss of randomness from having only 8 choices of j. Further loss from non-randomness of Hamming weights: Hamming weights around 66 are much more likely than at the edges; effect still noticeable after reduction to 8 choices.

Our analysis shows that the total loss is 6.9993%. This loss is justified by the very fast iteration function.

Average number of iterations for our attack against ECC2K-130: $\sqrt{\pi\ell/(2\cdot 2\cdot 131)} \cdot 1.069993 \approx 2^{60.9}$.

Field arithmetic required

Look more closely at costs of iteration function:

- one normal-basis Hamming-weight computation;
- one application of σ^j for some j ∈ {3,4,...,10}:
 ≤ 20S if computed as a series of squarings;

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one elliptic-curve addition:
 1I + 2M + 1S + 7a in affine coordinates.

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 ≤ 20S if computed as a series of squarings;
- one elliptic-curve addition:
 1I + 2M + 1S + 7a in affine coordinates.

"Montgomery's trick": handle N iterations in parallel; batch NI into 1I + (3N - 3)M.

Summary: Each iteration costs $\leq (1/N)(I - 3M) + 5M + 21S + 7a$ plus a Hamming-weight computation in normal basis.

How to perform these operations most efficiently?

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Bit operations

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We can compute an iteration using a straight-line (branchless) sequence of 70467 + 70263/N two-input bit operations.
e.g. 71880 bit operations/iteration for N = 51.
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Bit operations: "AND" and "XOR";

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Compare to 34061 bit operations $(131^2 \text{ ANDs} + 130^2 \text{ XORs})$ for one schoolbook multiplication of two 131-bit polynomials.

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Compare to 34061 bit operations $(131^2 \text{ ANDs} + 130^2 \text{ XORs})$ for one schoolbook multiplication of two 131-bit polynomials.

Fortunately, there are faster multiplication methods. http://binary.cr.yp.to/m.html: $M(131) \leq 11961$ where M(n) is minimum # bit operations for *n*-bit multiplication.

We could use the polynomial basis $1, z, z^2, \ldots, z^{130}$ of $F_{2^{131}} = F_2[z]/(z^{131} + z^{13} + z^2 + z + 1).$

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Or we could use the "type-2 optimal normal basis" $\zeta + 1/\zeta, \zeta^2 + 1/\zeta^2, \zeta^4 + 1/\zeta^4, \ldots, \zeta^{2^{130}} + 1/\zeta^{2^{130}}$ where ζ is a primitive 263rd root of 1.

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Well-known advantages of normal basis:

- ► The 21**S** are free.
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Surprise: Our best results use normal basis!

The Shokrollahi multiplier

2007 Shokrollahi, von zur Gathen-Shokrollahi-Shokrollahi: Can convert from a length-*n* type-2 optimal normal basis $\zeta + 1/\zeta, \zeta^2 + 1/\zeta^2, \zeta^4 + 1/\zeta^4, \ldots$ to $1, \zeta + 1/\zeta, (\zeta + 1/\zeta)^2, (\zeta + 1/\zeta)^3, \ldots$ using $\approx (1/2)(n \lg n)$ bit operations; similar for inverse. $\approx M(n+1) + 2n \lg n$ bit operations for normal-basis mult.

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New: Save bit operations by streamlining the conversion. M(131) + 1559 for size-131 normal-basis multiplication.

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New: Save bit operations by streamlining the conversion. M(131) + 1559 for size-131 normal-basis multiplication.

Save even more bit operations by mixing type-2 optimal normal basis, type-2 optimal polynomial basis. $\approx M(n) + n \lg n$ to multiply; M(131) + 917 for n = 131. $\approx (1/2)n \lg n$ before and after squarings; 325 for n = 131. For more details: 2009 Bernstein-Lange, forthcoming.

Bit operations vs. reality

Conventional wisdom: Counting bit operations is too simple. Analyzing and optimizing performance on Phenom II, Core 2, GTX 295, Cell, XC3S5000, etc. is much more work:

- Natural units are words (32 bits or more), not bits.
- ► Not limited to AND, XOR. Can use, e.g., array lookups.
- Extracting individual bits is feasible but relatively slow.
- ▶ Need to copy code into small "cache". Copies are slow.
- Need to copy data into small "cache"—and then into tiny "register set". ("Load" into register; "store" from register.)
- On "two-operand" CPUs, each arithmetic operation overwrites one of its input registers.
- Can perform several *independent* operations at once.

Example: The Cell implementation(s)

Cell teams: Joppe Bos and Thorsten Kleinjung, Lausanne; Peter Schwabe, Eindhoven, later joined by Ruben Niederhagen.

Tried two different implementation strategies for the Cell:

traditional "non-bitsliced" (using polynomial basis);

"bitsliced" (polynomial basis at first, then normal basis).

Cell inside PlayStation 3 has 6 accessible "SPU" cores. Each core runs at 3.2GHz: i.e., 3.2 billion cycles/second. \leq 1 arithmetic operation/cycle. \leq 1 load or store/cycle. Large register set: 128 registers, each 128 bits. Small memory on each core: 256 kilobytes.

▶ 31 Jul: 2565 (non-bitsliced)

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- 31 Jul: 2565 (non-bitsliced)
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- 31 Jul: 2565 (non-bitsliced)
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13	Oct:	871	(bitsliced)
14	Oct:	844	(bitsliced)
15	Oct:	789	(bitsliced)
29	Oct:	749	(bitsliced)

f0 = 1;f1 = 0;g0 = 1;g1 = 1;c = f0 & g1;d = f1 & g0;h0 = f0 & g0; $h1 = c \uparrow d;$ h2 = f1 & g1;

5 bit operations.

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f0 = 1;	f0 = 1;
f1 = 0;	f1 = 1;
g0 = 1;	g0 = 0;
g1 = 1;	g1 = 1;
c = f0 & g1;	c = f0 & g1;
d = f1 & g0;	d = f1 & g0;
h0 = f0 & g0;	h0 = f0 & g0
$h1 = c \cap d;$	$h1 = c \uparrow d;$
h2 = f1 & g1;	h2 = f1 & g1

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5 bit operations. 5 bit operations.

f0 = 1;	f0 = 1;	f0 = 0;
f1 = 0;	f1 = 1;	f1 = 1;
g0 = 1;	g0 = 0;	g0 = 0;
g1 = 1;	g1 = 1;	g1 = 1;
<pre>c = f0 & g1;</pre>	<pre>c = f0 & g1;</pre>	<pre>c = f0 & g1;</pre>
d = f1 & g0;	d = f1 & g0;	d = f1 & g0;
h0 = f0 & g0;	h0 = f0 & g0;	h0 = f0 & g0;
h1 = c ^ d;	h1 = c ^ d;	h1 = c ^ d;
h2 = f1 & g1;	h2 = f1 & g1;	h2 = f1 & g1;

5 bit operations.

5 bit operations

5 bit operations.

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f0 = bitvector(1,1,0); f1 = bitvector(0,1,1); g0 = bitvector(1,0,0); g1 = bitvector(1,1,1);

5 vector operations.

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Bitslicing disadvantages:

- Table lookups such as tab[f mod 16] are expensive.
- Conditional branches are expensive.
- 128× volume of data (assuming 128-bit vectors); harder to avoid load/store bottlenecks.
- Transposition costs roughly 1 cycle per byte; frequent transposition is bad.

Bitslicing advantages:

- ► Free bit extraction, bit shuffling, etc.
- No word-size penalty. Example: 128 sums of *d*-bit polynomials cost *d* vector xors instead of 128[*d*/128]. Huge speedup for small *d*.
- Productive synergy with M(n) techniques.

2466 Cell CPUs for a year: i.e., 900000 machine days. Recall Certicom's estimate: 2700000000 machine days.

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"Computers ten years ago didn't do much work per cycle!"

2466 Cell CPUs for a year: i.e., 900000 machine days. Recall Certicom's estimate: 2700000000 machine days.

"That's unfair! Computers ten years ago were very slow!" Indeed, Certicom's "machine" was a 100MHz Pentium. Today's Cell has several cores, each running at 3.2GHz. Scale by counting cycles ... and we're still 15× faster.

"Computers ten years ago didn't do much work per cycle!" True for the Pentium ... but not for Harley's Alpha, which had 64-bit registers, 4 instructions/cycle, etc.

Harley's ECC2K-108 software uses 1651 Alpha cycles/iteration. We ran the same software on a Core 2: 1800 cycles/iteration. We also wrote our own polynomial-basis ECC2K-108 software: on the same Core 2, fewer than 500 cycles/iteration.

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Eight central servers (at TU Eindhoven) receive points, pre-sort the points into 8192 RAM buffers, flush the buffers to 8192 disk files.

Periodically read each file into RAM, sort, find collisions. Also double-check random samples for validity.

Written to disk so far: 6 gigabytes. Reading, sorting, finding collisions: 10 seconds per server.

Client-server communication

Each report is compressed to 8-byte seed, 8-byte hash. Negligible cost for server to recompute distinguished point.

We have clusters computing points at several sites worldwide. Currently 2608 separate streams of data entering servers.

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Lightweight data-transfer protocol:

 Client collects 64 reports into a 1024-byte block; adds 8 bytes identifying client site, stream, block position; sends resulting packet to server through UDP.

Server has 4-byte stream state, sends back 8-byte ack.

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Each packet is encrypted, authenticated, verified, decrypted using http://nacl.cace-project.eu; costs 16 bytes. Total block cost: 1090-byte IP packet plus 66-byte ack.

Clients so far

Current tally of points reported to servers from various sites:

- ▶ site "ℓ": 4329046016 bytes
- ▶ site "L": 2558473216 bytes
- ▶ site "G": 768631808 bytes
- site "j": 305588224 bytes
- site "e": 269759488 bytes
- ▶ site "t": 204332032 bytes
- ▶ site "b": 123914240 bytes
- ▶ site "d": 31168512 bytes
- site "z": 1055744 bytes

Mix of Cell (PlayStations, blades), Core 2, Phenom, Working on collecting more clusters, building FPGA clusters, continuing to speed up the implementations (especially GPU).

Get more details, and watch our progress!

http://eprint.iacr.org/2009/466:

"The Certicom challenges ECC2-X" (SHARCS 2009) analysis of ECC2K-130, ECC2-131, ECC2K-163, ECC2-163 with ASIC, FPGA, Cell, Core2 implementation details.

http://eprint.iacr.org/2009/541:

"Breaking ECC2K-130"; continues to be improved; more platforms, better speeds, running the attack.

http://ecc-challenge.info:

anonymous web page (but ours, really!), including graph of number of points reported to the servers.

https://twitter.com/ECCchallenge: anonymous Twitter page with the latest announcements.

Hope to finish attack in first half of 2010.