The factorization of RSA-1024

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Abstract: This talk discusses the most important tools for attackers breaking 1024-bit RSA keys today and tomorrow. The same tools will also be useful for academic teams in the farther future publicly breaking the RSA-1024 challenge.

Sieving small integers $i > 0$
using primes $2, 3, 5, 7$:

1				
2	2			
3		3		
4	2	2		
5			5	
6	2		3	
7				7
8	2	2	2	
9			3	3
10	2			5
11				
12	2	2	3	
13				
14	2			7
15			3	5
16	2	2	2	2
17				
18	2		3	3
19				
20	2	2		5

etc.

Sieving i and $611 + i$ for small i using primes $2, 3, 5, 7$:

1	
2	2
3	3
4	2 2
5	5
6	2 3
7	7
8	2 2 2
9	3 3
10	2 5
11	
12	2 2 3
13	
14	2 7
15	3 5
16	2 2 2 2
17	
18	2 3 3
19	
20	2 2 5

612	2 2 3 3
613	
614	2
615	3 5
616	2 2 2 7
617	
618	2 3
619	
620	2 2 5
621	3 3 3
622	2
623	7
624	2 2 2 2 3
625	5 5 5 5
626	2
627	3
628	2 2
629	
630	2 3 3 5 7
631	

etc.

Have complete factorization of the "congruences" $i(611+i)$ for some i's.

$14 \cdot 625 = 2^1 3^0 5^4 7^1$.
$64 \cdot 675 = 2^6 3^3 5^2 7^0$.
$75 \cdot 686 = 2^1 3^1 5^2 7^3$.

$14 \cdot 64 \cdot 75 \cdot 625 \cdot 675 \cdot 686$
$= 2^8 3^4 5^8 7^4 = (2^4 3^2 5^4 7^2)^2$.

$\gcd\{611, 14 \cdot 64 \cdot 75 - 2^4 3^2 5^4 7^2\}$
$= 47$.

$611 = 47 \cdot 13$.

Why did this find a factor of 611?
Was it just blind luck:
$\gcd\{611, \text{random}\} = 47$?

No.

By construction 611 divides $s^2 - t^2$
where $s = 14 \cdot 64 \cdot 75$
and $t = 2^4 3^2 5^4 7^2$.
So each prime > 7 dividing 611
divides either $s - t$ or $s + t$.

Not terribly surprising
(but not guaranteed in advance!)
that one prime divided $s - t$
and the other divided $s + t$.

Why did the first three completely factored congruences have square product?
Was it just blind luck?

Yes. The exponent vectors $(1,0,4,1), (6,3,2,0), (1,1,2,3)$ happened to have sum 0 mod 2.

But we didn't need this luck!
Given long sequence of vectors, easily find nonempty subsequence with sum 0 mod 2.

This is linear algebra over $\mathbf{F}_2$.
Guaranteed to find subsequence
if number of vectors
exceeds length of each vector.

e.g. for $n = 671$:

$1(n+1) = 2^5 3^1 5^0 7^1$;
$4(n+4) = 2^2 3^3 5^2 7^0$;
$15(n+15) = 2^1 3^1 5^1 7^3$;
$49(n+49) = 2^4 3^2 5^1 7^2$;
$64(n+64) = 2^6 3^1 5^1 7^2$.

$\mathbf{F}_2$-kernel of exponent matrix is
gen by (0 1 0 1 1) and (1 0 1 1 0);
e.g., $1(n+1)15(n+15)49(n+49)$
is a square.

Plausible conjecture: $\mathbf{Q}$ sieve can separate the odd prime divisors of any n, not just 611.

Given n and parameter y:

Try to completely factor $i(n+i)$ for $i \in \{1, 2, 3, \ldots, y^2\}$ into products of primes $\leq y$.

Look for nonempty set of i's with $i(n+i)$ completely factored and with $\prod_i i(n+i)$ square.

Compute $\gcd\{n, s-t\}$ where $s = \prod_i i$ and $t = \sqrt{\prod_i i(n+i)}$.

Generalizing beyond $\mathbf{Q}$

The $\mathbf{Q}$ sieve is a special case of the number-field sieve (NFS).

Recall how the $\mathbf{Q}$ sieve factors 611:

Form a square as product of $i(i + 611j)$ for several pairs (i, j):
$14(625) \cdot 64(675) \cdot 75(686)$
$= 4410000^2$.

$\gcd\{611, 14 \cdot 64 \cdot 75 - 4410000\}$
$= 47$.

The $\mathbf{Q}(\sqrt{14})$ sieve
factors 611 as follows:

Form a square
as product of $(i + 25j)(i + \sqrt{14}j)$
for several pairs (i, j):

$$(-11 + 3 \cdot 25)(-11 + 3\sqrt{14})$$
$$\cdot (3 + 25)(3 + \sqrt{14})$$
$$= (112 - 16\sqrt{14})^2.$$

Compute
$s = (-11 + 3 \cdot 25) \cdot (3 + 25)$,
$t = 112 - 16 \cdot 25$,
$\gcd\{611, s - t\} = 13$.

Why does this work?

Answer: Have ring morphism $\mathbf{Z}[\sqrt{14}] \to \mathbf{Z}/611$, $\sqrt{14} \mapsto 25$, since $25^2 = 14$ in $\mathbf{Z}/611$.

Apply ring morphism to square:

$$(-11 + 3 \cdot 25)(-11 + 3 \cdot 25) \cdot (3 + 25)(3 + 25)$$
$$= (112 - 16 \cdot 25)^2 \text{ in } \mathbf{Z}/611.$$

i.e. $s^2 = t^2$ in $\mathbf{Z}/611$.

Unsurprising to find factor.

Generalize from $(x^2 - 14, 25)$
to (f, m) with irred $f \in \mathbf{Z}[x]$,
$m \in \mathbf{Z}$, $f(m) \in n\mathbf{Z}$.

Write $d = \deg f$,
$f = f_d x^d + \cdots + f_1 x^1 + f_0 x^0$.

Can take $f_d = 1$ for simplicity,
but larger f_d allows
better parameter selection.

Pick $\alpha \in \mathbf{C}$, root of f.
Then $f_d \alpha$ is a root of
monic $g = f_d^{d-1} f(x/f_d) \in \mathbf{Z}[x]$.

$$\mathbf{Q}(\alpha) = \left\{ \begin{array}{l} r_0 + r_1\alpha + r_2\alpha^2 + \\ \cdots + r_{d-1}\alpha^{d-1}: \\ r_0, \ldots, r_{d-1} \in \mathbf{Q} \end{array} \right\}$$

$$\uparrow$$

$$\mathcal{O} = \left\{ \begin{array}{c} \text{algebraic integers} \\ \text{in } \mathbf{Q}(\alpha) \end{array} \right\}$$

$$\uparrow$$

$$\mathbf{Z}[f_d\alpha] = \left\{ \begin{array}{l} i_0 + i_1 f_d\alpha + \\ \cdots + i_{d-1} f_d^{d-1}\alpha^{d-1}: \\ i_0, \ldots, i_{d-1} \in \mathbf{Z} \end{array} \right\}$$

$$\downarrow \; f_d\alpha \mapsto f_d m$$

$$\mathbf{Z}/n = \{0, 1, \ldots, n-1\}$$

Build square in $\mathbf{Q}(\alpha)$ from congruences $(i - jm)(i - j\alpha)$ with $i\mathbf{Z} + j\mathbf{Z} = \mathbf{Z}$ and $j > 0$.

Could replace $i - jx$ by higher-deg irred in $\mathbf{Z}[x]$; quadratics seem fairly small for some number fields.
But let's not bother.

Say we have a square $\prod_{(i,j)\in S}(i - jm)(i - j\alpha)$ in $\mathbf{Q}(\alpha)$; now what?

$\prod(i - jm)(i - j\alpha)f_d^2$
is a square in $\mathcal{O}$,
ring of integers of $\mathbf{Q}(\alpha)$.

Multiply by $g'(f_d\alpha)^2$,
putting square root into $\mathbf{Z}[f_d\alpha]$:
compute r with $r^2 = g'(f_d\alpha)^2 \cdot$
$\prod(i - jm)(i - j\alpha)f_d^2$.

Then apply the ring morphism
$\varphi : \mathbf{Z}[f_d\alpha] \to \mathbf{Z}/n$ taking
$f_d\alpha$ to $f_d m$. Compute $\gcd\{n,$
$\varphi(r) - g'(f_d m)\prod(i - jm)f_d\}$.
In $\mathbf{Z}/n$ have $\varphi(r)^2 =$
$g'(f_d m)^2 \prod(i - jm)^2 f_d^2$.

How to find square product
of congruences $(i - jm)(i - j\alpha)$?

Start with congruences for,
e.g., y^2 pairs (i, j).

Look for y-smooth congruences:
y-smooth $i - jm$ and
y-smooth $f_d \operatorname{norm}(i - j\alpha) =$
$f_d i^d + \cdots + f_0 j^d = j^d f(i/j)$.
Here "y-smooth" means
"has no prime divisor $> y$."

Find enough smooth congruences.
Perform linear algebra on
exponent vectors mod 2.

Optimizing NFS

Finding smooth congruences
is *always* a bottleneck.

"What if it's much faster
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Answer: If it is, trivially
save time by decreasing y.

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My main focus today:
speed of smoothness detection.

Not covered in this talk:
optimizing choice of f,
set of pairs (i, j), etc.

1977 Schroeppel "linear sieve," forerunner of QS and NFS:
Factor $n \approx s^2$ using congruences $(s+i)(s+j)((s+i)(s+j)-n)$.
Sieve these congruences.

1996 Pomerance:
"The time for doing this is unbelievably fast compared with trial dividing each candidate number to see if it is Y-smooth. If the length of the interval is N, the number of steps is only about $N \log\log Y$, or about $\log\log Y$ steps on average per candidate."

Fact: These simple "steps"
become very slow as y increases.
Distant RAM is very slow.

Sieving small primes isn't bad,
but sieving large primes is
much slower than arithmetic.

Every recent NFS record
actually uses other methods
to find large primes:
e.g., SQUFOF, $p-1$, ECM.

For optimized RSA-1024 NFS,
ECM is the most important
step in smoothness detection.

ECM speedup team:

1	2	3	4	Daniel J. Bernstein
1	2	3	4	Tanja Lange
1			4	Peter Birkner
1				Christiane Peters
	2	3		Chen-Mou Cheng
	2	3		Bo-Yin Yang
	2			Tien-Ren Chen
		3		Hsueh-Chung Chen
		3		Ming-Shing Chen
		3		Chun-Hung Hsiao
		3		Zong-Cing Lin

1. “ECM using Edwards curves.”
Prototype software: GMP-EECM.
New rewrite: EECM-MPFQ.

2. “ECM on graphics cards.”
Prototype CUDA-EECM.

3. “The billion-mulmod-per-second PC.”
Current CUDA-EECM,
plus fast mulmods on
Core 2, Phenom II, and Cell.

4. “Starfish on strike.”
Integrated into EECM-MPFQ.

5. Not covered in this talk:
early-abort ECM optimization.

Fewer mulmods per curve

Measurements of EECM-MPFQ for $B_1 = 1000000$:

$b = 1442099$ bits in
$s = \mathrm{lcm}\{1, 2, 3, 4, \ldots, B_1\}$.

$P \mapsto sP$ is computed using
1442085 ($= 0.99999b$) DBL +
98341 ($0.06819b$) ADD.

These DBLs and ADDs use
5112988**M** ($3.54552b$**M**) +
5768340**S** ($3.99996b$**S**) +
9635920**add** ($6.68187b$**add**).

Compare to GMP-ECM 6.2.3:

$P \mapsto sP$ is computed using
2001915 (1.38820b) DADD +
194155 (0.13463b) DBL.

These DADDs and DBLs use
8590140**M** (5.95669b**M**) +
4392140**S** (3.04566b**S**) +
12788124**add** (8.86772b**add**).

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Could do better! 0.13463b**M**
are actually 0.13463b**D**.
D: mult by curve constant.
Small curve, small P, ladder
$\Rightarrow 4b\mathbf{M} + 4b\mathbf{S} + 2b\mathbf{D} + 8b\mathbf{add}$.
EECM still wins.

HECM handles 2 curves using $2b\mathbf{M} + 6b\mathbf{S} + 8b\mathbf{D} + \cdots$ (1986 Chudnovsky–Chudnovsky, et al.); again EECM is better.

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What about NFS? $B_1 = 587$?
Measurements of EECM-MPFQ:

$b = 839$ bits in s.

$P \mapsto sP$ is computed using
833 $(0.99285b)$ DBL $+$
131 $(0.15614b)$ ADD.

These DBLs and ADDs use
3552**M** $(4.23361b\mathbf{M}) +$
3332**S** $(3.97139b\mathbf{S}) +$
6308**add** $(7.51847b\mathbf{add})$.

Note: smaller window size
in addition chain,
so more ADDs per bit.

Compare to GMP-ECM 6.2.3:

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Compare to GMP-ECM 6.2.3:

$P \mapsto sP$ is computed using
4785**M** ($5.70322b$**M**) +
2495**S** ($2.97378b$**S**) +
7053**add** ($8.40644b$**add**).

Even for this small B_1,
EECM beats Montgomery ECM
in operation count.

Notes on current stage 2:

1. EECM-MPFQ jumps through the j's coprime to d_1.
GMP-ECM: coprime to 6.

2. EECM-MPFQ computes Dickson polynomial values using Bos–Coster addition chains.
GMP-ECM: ad-hoc, relying on arithmetic progression of j.

3. EECM-MPFQ doesn't bother converting to affine coordinates until the end of stage 2.

4. EECM-MPFQ uses NTL for poly arith in "big" stage 2.

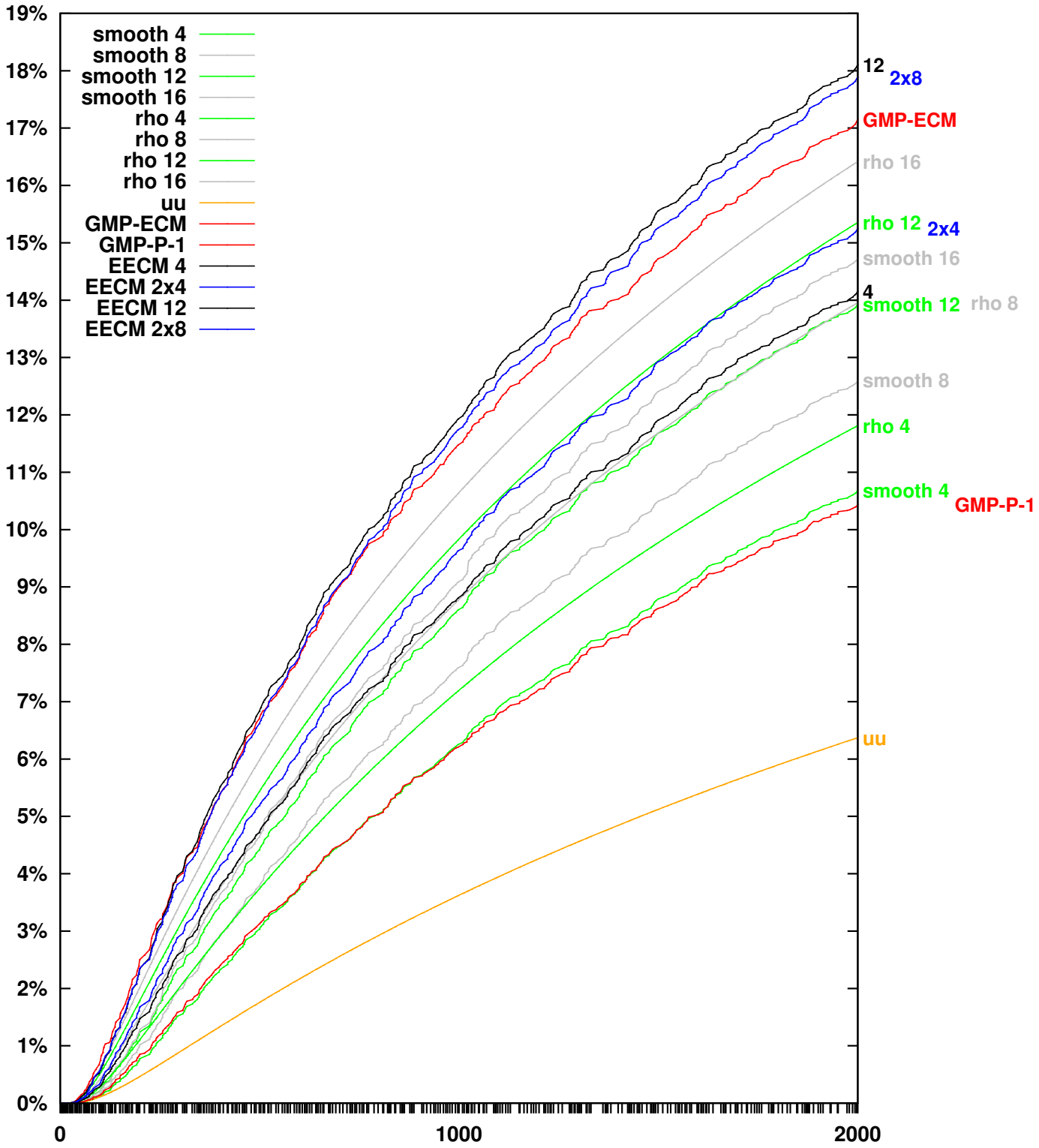
More primes per curve

1987/1992 Montgomery,
1993 Atkin–Morain
had suggested using torsion
$\mathbf{Z}/12$ or $(\mathbf{Z}/2) \times (\mathbf{Z}/8)$.

GMP-ECM went back to $\mathbf{Z}/6$.

"ECM using Edwards curves"
introduced new small curves
with $\mathbf{Z}/12$, $(\mathbf{Z}/2) \times (\mathbf{Z}/8)$.

Does big torsion really help?
Let's try a random sample
of 65536 30-bit primes.


19%
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2%
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smooth 4
smooth 8
smooth 12
smooth 16
rho 4
rho 8
rho 12
rho 16
uu
GMP-ECM
GMP-P-1
EECM 4
EECM 2x4
EECM 12
EECM 2x8
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2x8
GMP-ECM
rho 16
rho 12
2x4
smooth 16
4
smooth 12
rho 8
smooth 8
rho 4
smooth 4
GMP-P-1
uu

Fastest known ADDs are for $-x^2 + y^2 = 1 + dx^2y^2$, which can't have > 8 torsion points.

"Starfish on strike":
Is the sacrifice in torsion justified by the ADD speedup?

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Surprising phenomenon: $\mathbf{Z}/6$ $-x^2 + y^2 = 1 + dx^2y^2$ family finds *more* primes than $\mathbf{Z}/12$. Best ECM family known.

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Even more benefit from precomputing best curves.

Faster mulmods

ECM is bottlenecked by mulmods:

- practically all of stage 1;
- curve operations in stage 2 (pumped up by Dickson!);
- final product in stage 2, *except* fast poly arith.

GMP-ECM does mulmods with the GMP library.

. . . but GMP has slow API, so GMP-ECM has ≥ 20000 lines of new mulmod code.

```
$ wc -c<eecm-mpfq.tar.bz2
```

16031

Obviously EECM-MPFQ doesn't include new mulmod code!

```
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```

Obviously EECM-MPFQ doesn't include new mulmod code!

MPFQ library (Gaudry–Thomé) does arithmetic in $\mathbf{Z}/n$ where number of n words is known at compile time.
Better API than GMP: most importantly, n in advance.

EECM-MPFQ uses MPFQ for essentially all mulmods.

GMP-ECM 6.2.3/GMP 4.3.2:

Tried 1000 curves, $B_1 = 2000$,
typical 240-bit n,
on 3.2GHz Phenom II x4.

Stage 1: $7.4 \cdot 10^6$ cycles/curve.

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EECM-MPFQ,
same 240-bit n, same CPU,
1000 curves, $B_1 = 2000$:
$5.2 \cdot 10^6$ cycles/curve.

Some speedup from Edwards;
some speedup from MPFQ.

What about stage 2?

GMP-ECM, 1000 curves,
$B_1 = 587$, $B_2 = 15366$,
Dickson polynomial degree 1:
$6.6 \cdot 10^6$ cycles/curve.
Degree 3: $9.5 \cdot 10^6$.

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$6.6 \cdot 10^6$ cycles/curve.
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EECM-MPFQ, 1000 curves,
$B_1 = 587$, $d_1 = 420$, range 20160
for primes $420i \pm j$:
$2.6 \cdot 10^6$ cycles/curve.
Degree 3: $3.1 \cdot 10^6$.

Summary: EECM-MPFQ uses fewer mulmods than GMP-ECM; takes less time than GMP-ECM; and finds more primes.

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Are GMP-ECM and EECM-MPFQ fully exploiting the CPU? No!

Three recent efforts to speed up mulmods for ECM: Thorsten Kleinjung, for RSA-768; Alexander Kruppa, for CADO; and ours—see next slide.

Our latest mulmod speeds,
interleaving vector threads
with integer threads:

4×3GHz Phenom II 940:
$202 \cdot 10^6$ 192-bit mulmods/sec.

4×2.83GHz Core 2 Quad Q9550:
$114 \cdot 10^6$ 192-bit mulmods/sec.

6×3.2GHz Cell (Playstation 3):
$102 \cdot 10^6$ 195-bit mulmods/sec.

$500 GTX 295
is one card with two GPUs;
60 cores; 480 32-bit ALUs.
Runs at 1.242GHz.

Our latest CUDA-EECM speed:
$481 \cdot 10^6$ 210-bit mulmods/sec.

For $\approx$ $2000 can build PC
with one CPU and four GPUs:
$1300 \cdot 10^6$ 192-bit mulmods/sec.