Twisted Hessian curves

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University of Illinois at Chicago &
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Joint work with:

Chitchanok Chuengsatiansup
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David Kohel
Aix-Marseille Université

Tanja Lange
Technische Universiteit Eindhoven

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"The crucial problem becomes
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Most important computations:
ADD is $P, Q \mapsto P + Q$.
DBL is $P \mapsto 2P$.

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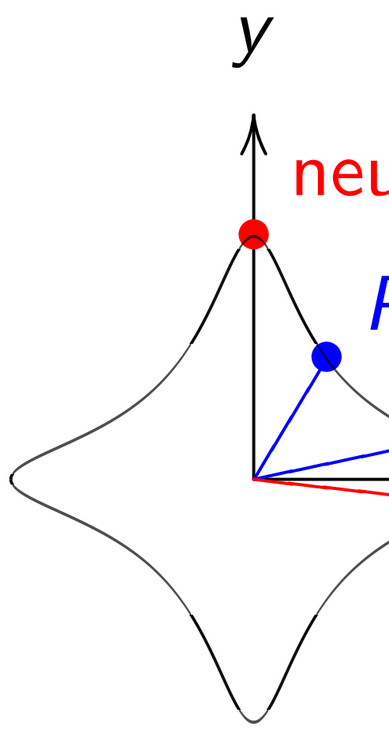

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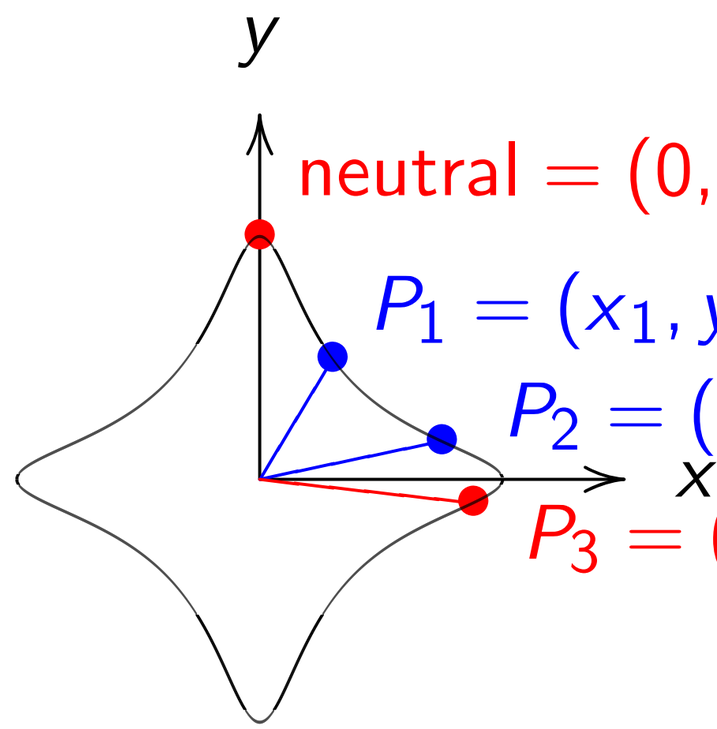

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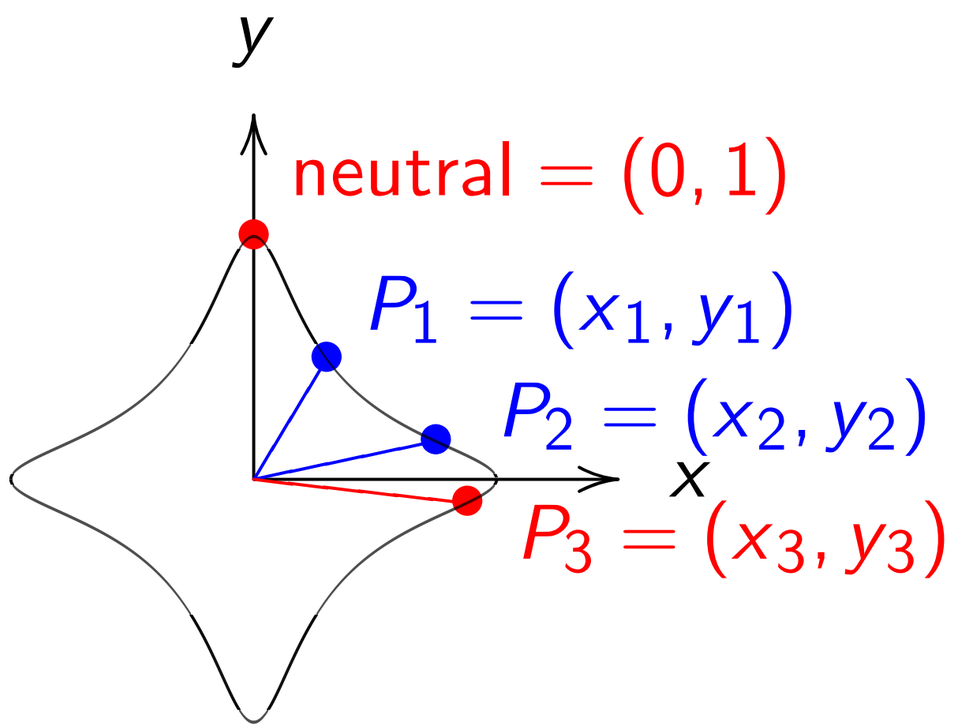

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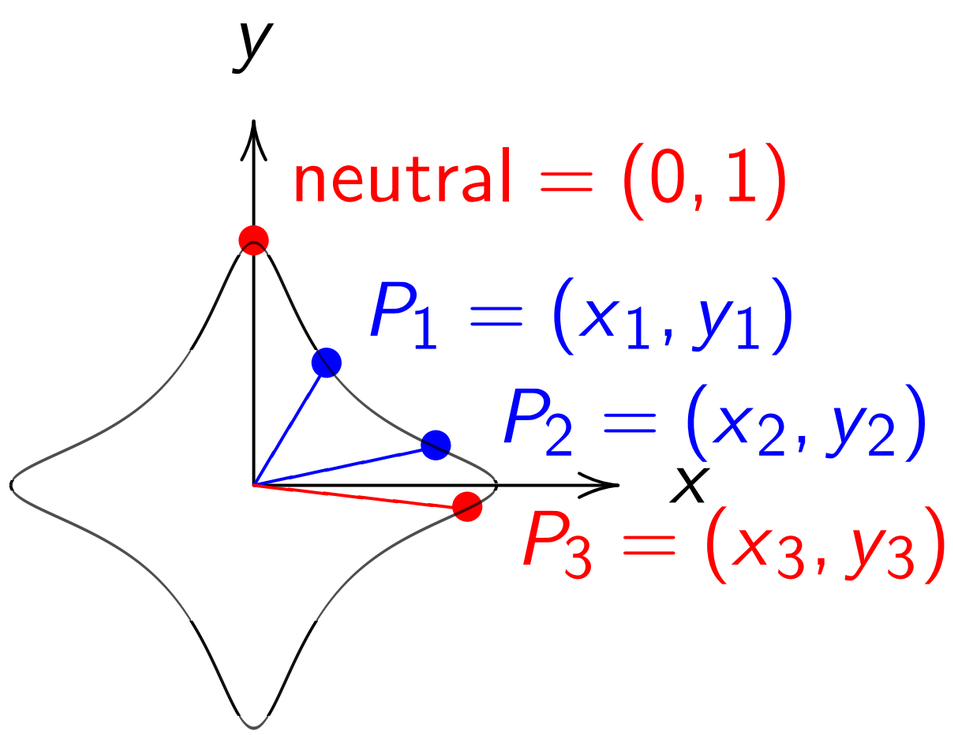

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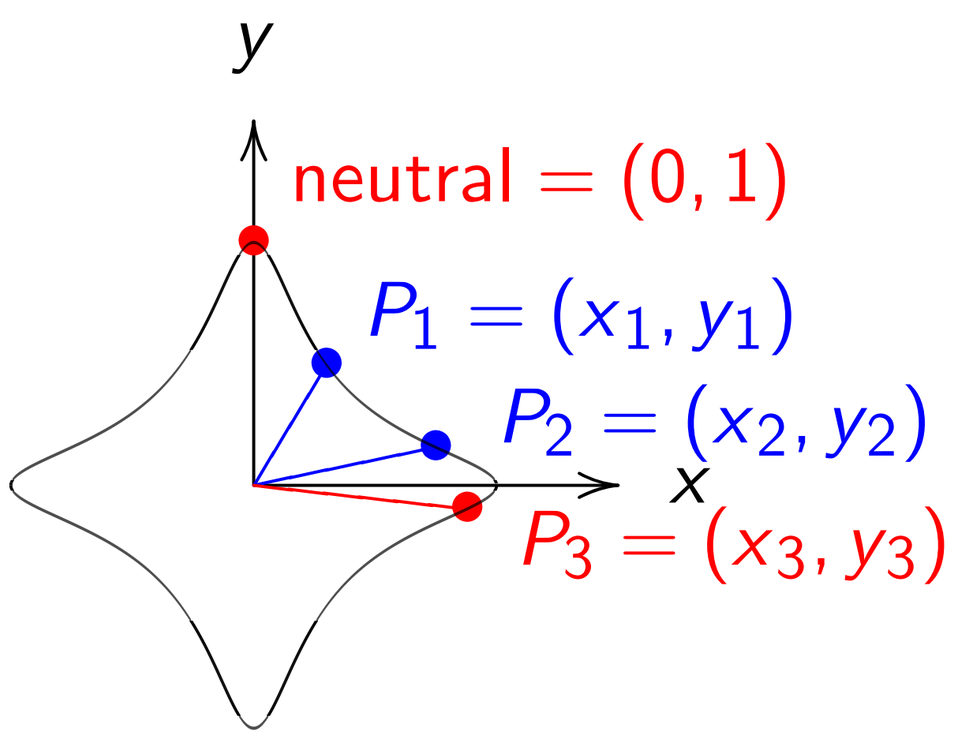

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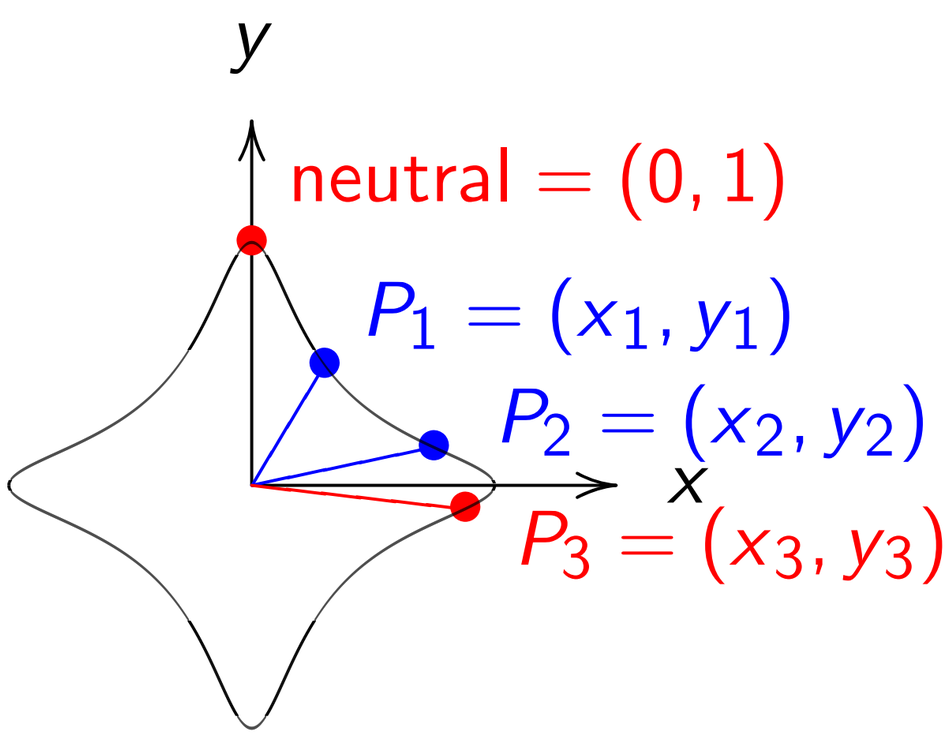

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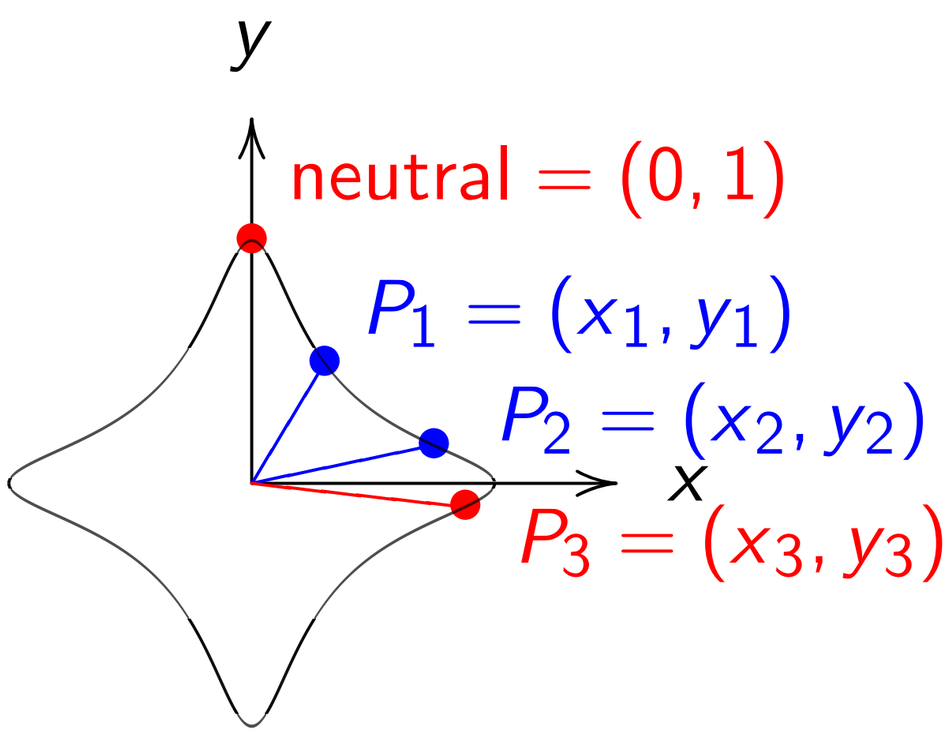

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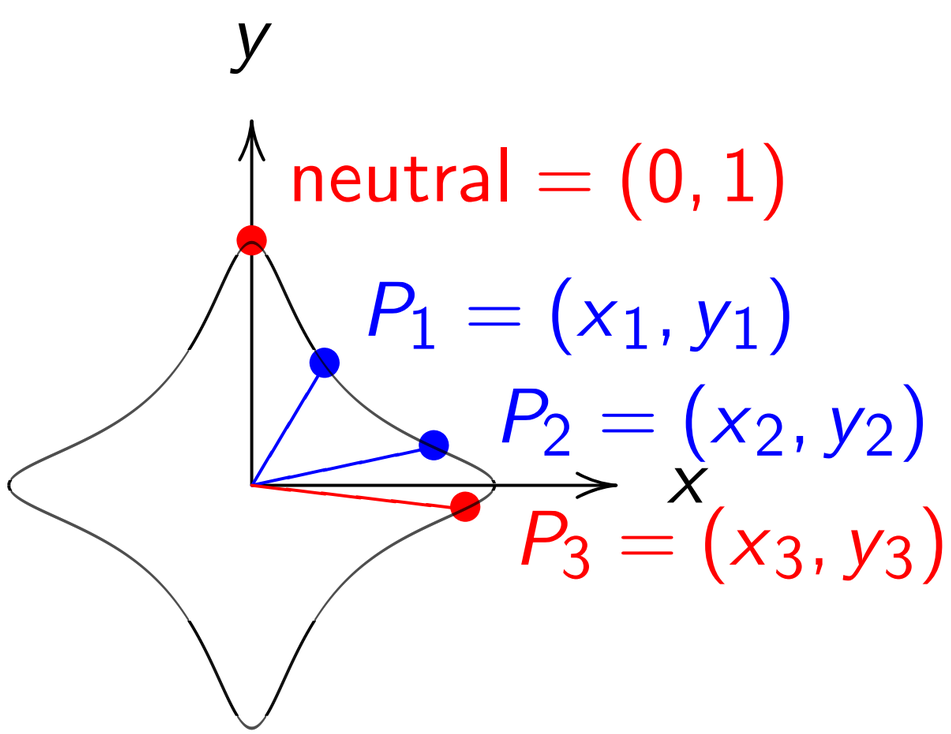
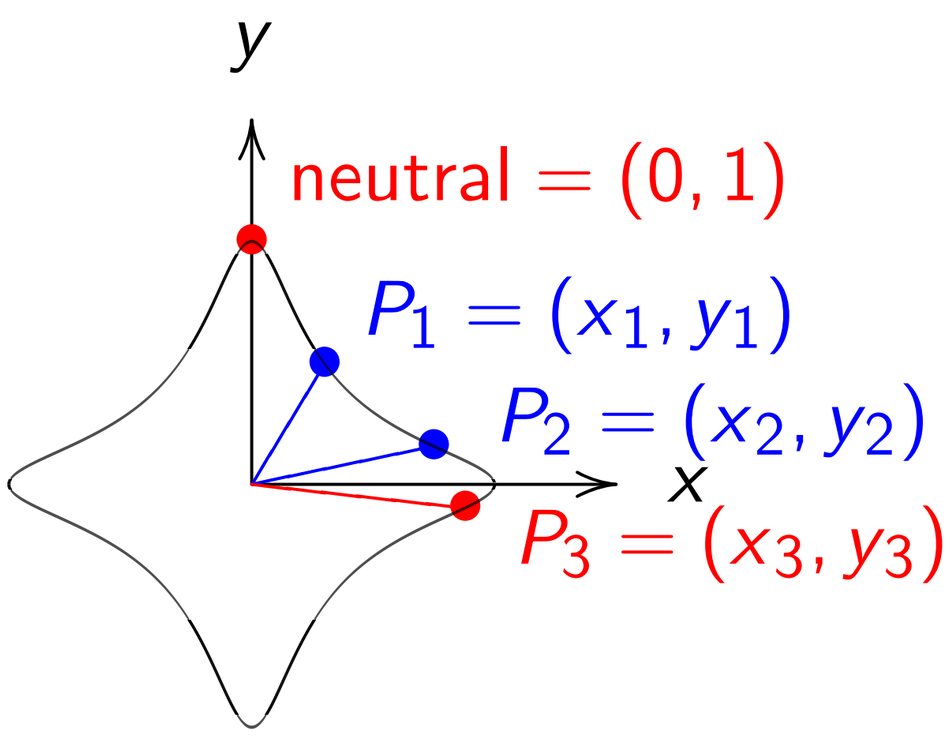


Example: $x^2 + y^2 = 1 - 30x^2y^2$.
Sum of (x_1, y_1) and (x_2, y_2) is
$((x_1y_2+y_1x_2)/(1-30x_1x_2y_1y_2)$,
$(y_1y_2-x_1x_2)/(1+30x_1x_2y_1y_2))$.

2007 Bernstein–Lange:
10.8**M** for ADD, 6.2**M** for DBL.

2008 Hisil–Wong–Carter–Dawson:
just 8**M** for ADD.

2007 Edwards: new curve shape.
2007 Bernstein–Lange: generalize,
analyze speed, completeness.



Example: $x^2 + y^2 = 1 - 30x^2y^2$.
Sum of (x_1, y_1) and (x_2, y_2) is
$((x_1y_2+y_1x_2)/(1-30x_1x_2y_1y_2)$,
$(y_1y_2-x_1x_2)/(1+30x_1x_2y_1y_2))$.

2007 Bernstein–Lange:
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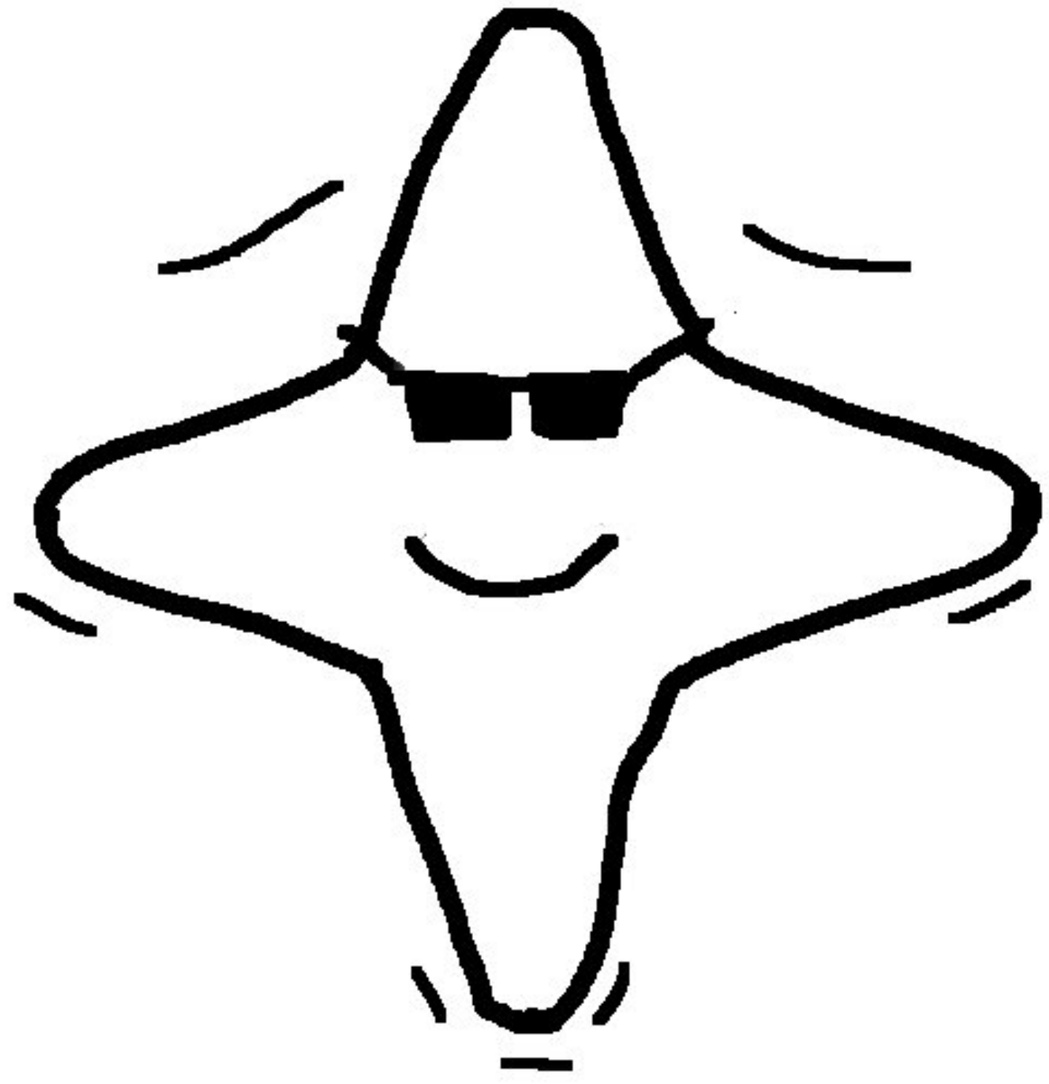
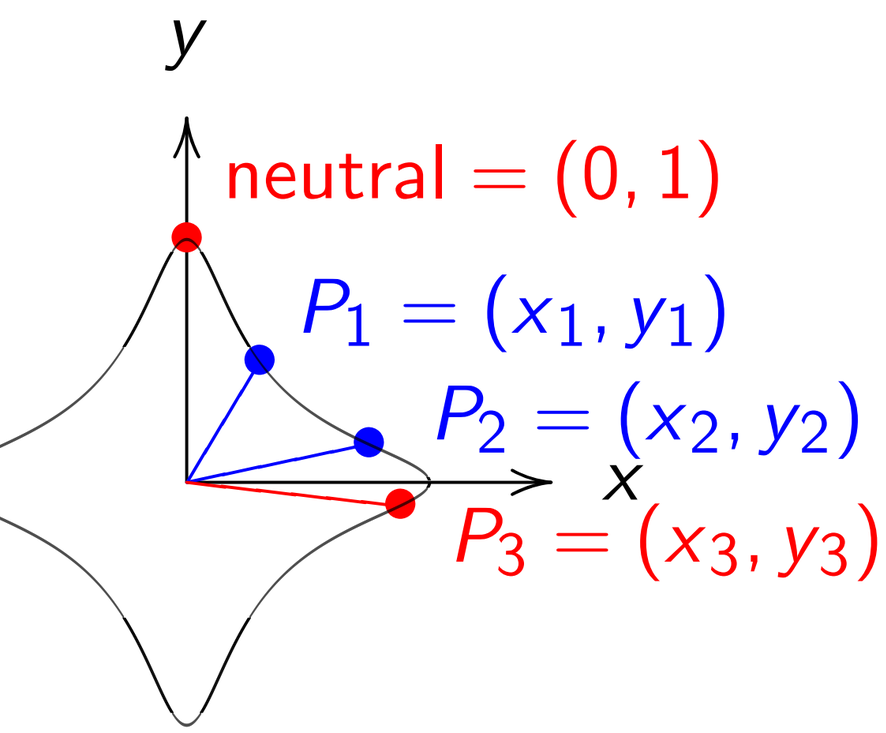


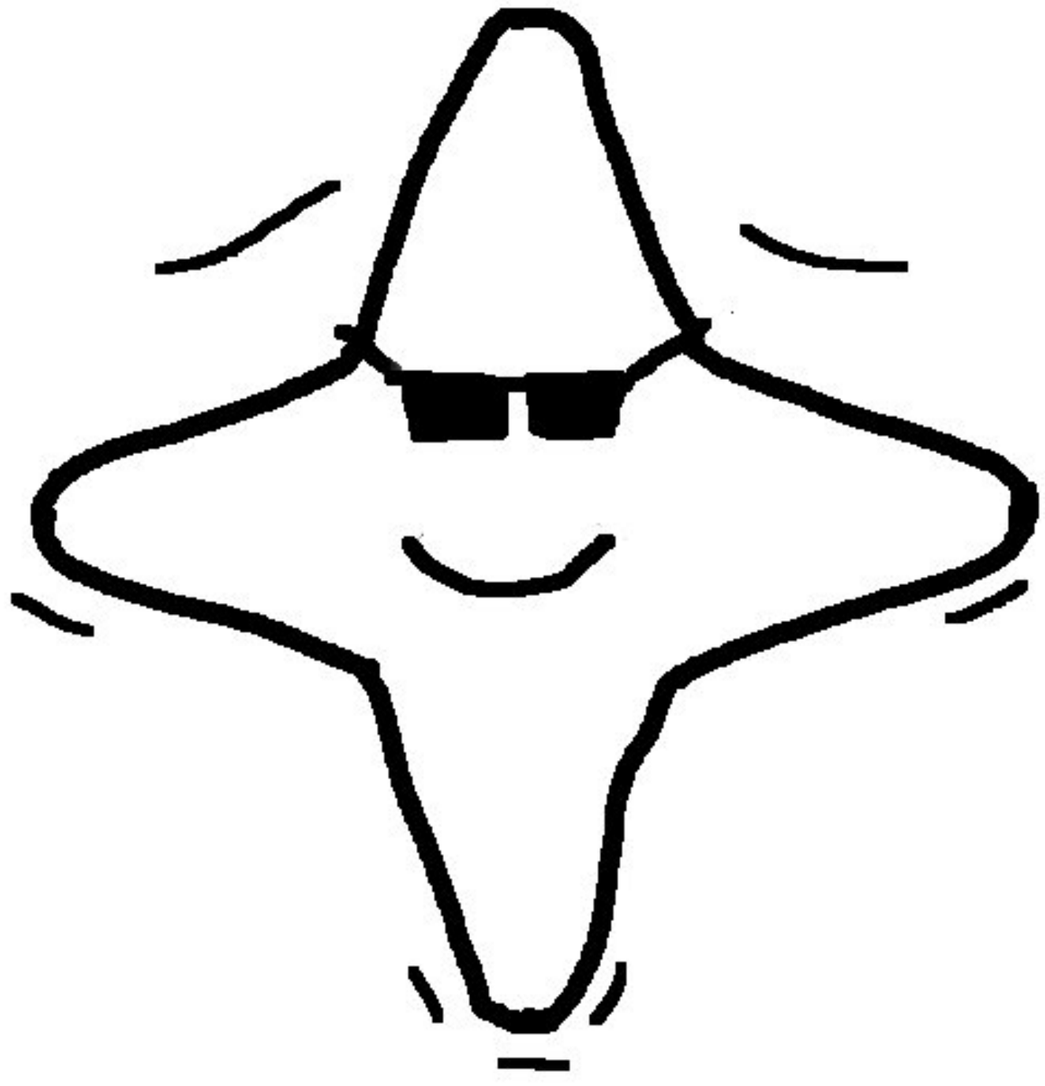
wards: new curve shape.
rnstein–Lange: generalize,
speed, completeness.



e: $x^2 + y^2 = 1 - 30x^2y^2$.
(x_1, y_1) and (x_2, y_2) is
$+y_1x_2)/(1-30x_1x_2y_1y_2)$,
$x_1x_2)/(1+30x_1x_2y_1y_2)$).

2007 Bernstein–Lange:
10.8**M** for ADD, 6.2**M** for DBL.

2008 Hisil–Wong–Carter–Dawson:
just 8**M** for ADD.



$y^2 = x^3$

w curve shape.
ange: generalize,
mpleteness.

utral $= (0, 1)$

$P_1 = (x_1, y_1)$

$P_2 = (x_2, y_2)$

x

$P_3 = (x_3, y_3)$

$= 1 - 30x^2y^2$.
nd (x_2, y_2) is
$-30x_1x_2y_1y_2)$,
$+30x_1x_2y_1y_2))$.

2007 Bernstein–Lange:
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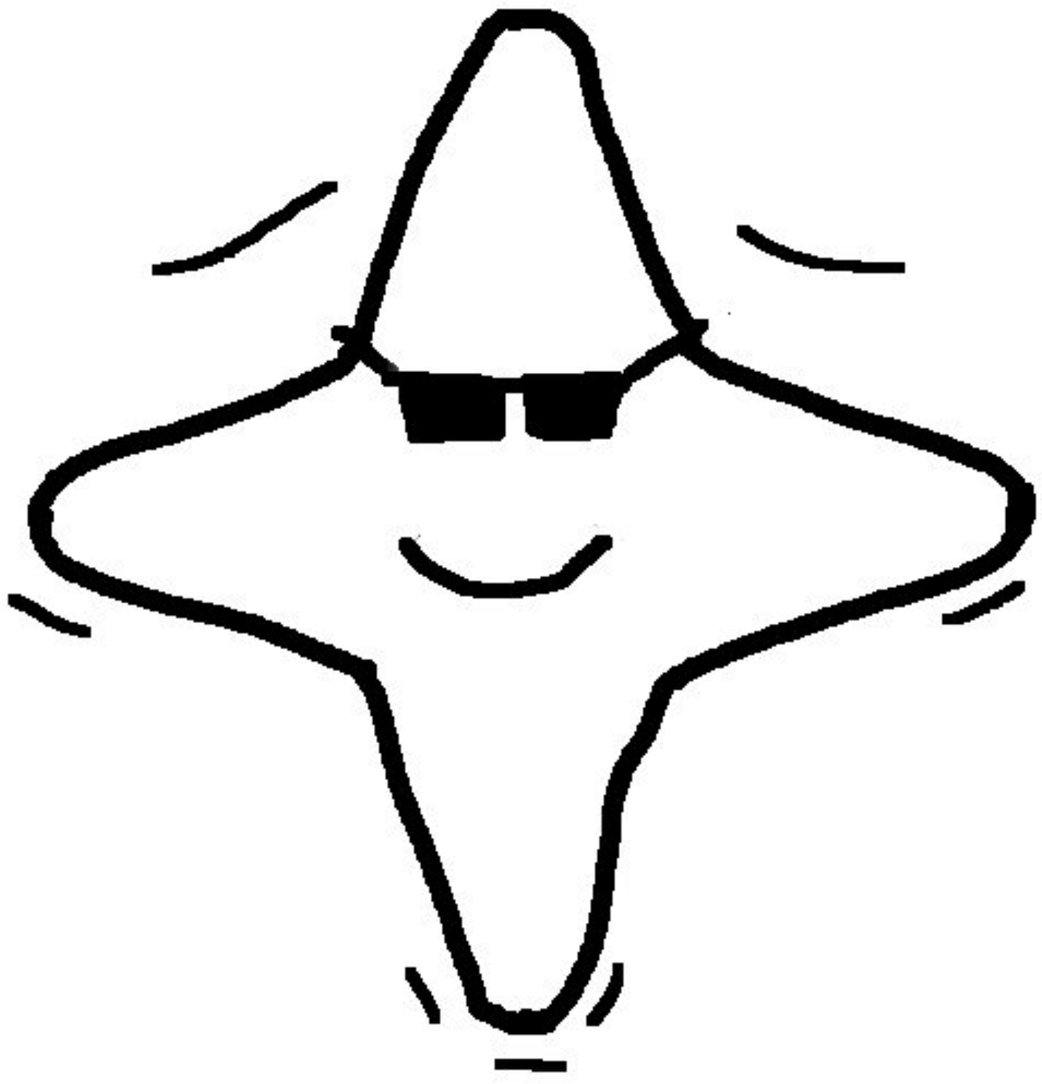

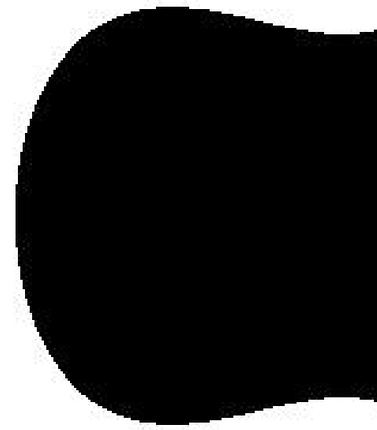

$y^2 = x^3 - 0.4x +$

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1)

$y_1)$

$x_2, y_2)$

(x_3, y_3)

x^2y^2.

) is

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$_1y_2)$).

2007 Bernstein–Lange:
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2008 Hisil–Wong–Carter–Dawson:
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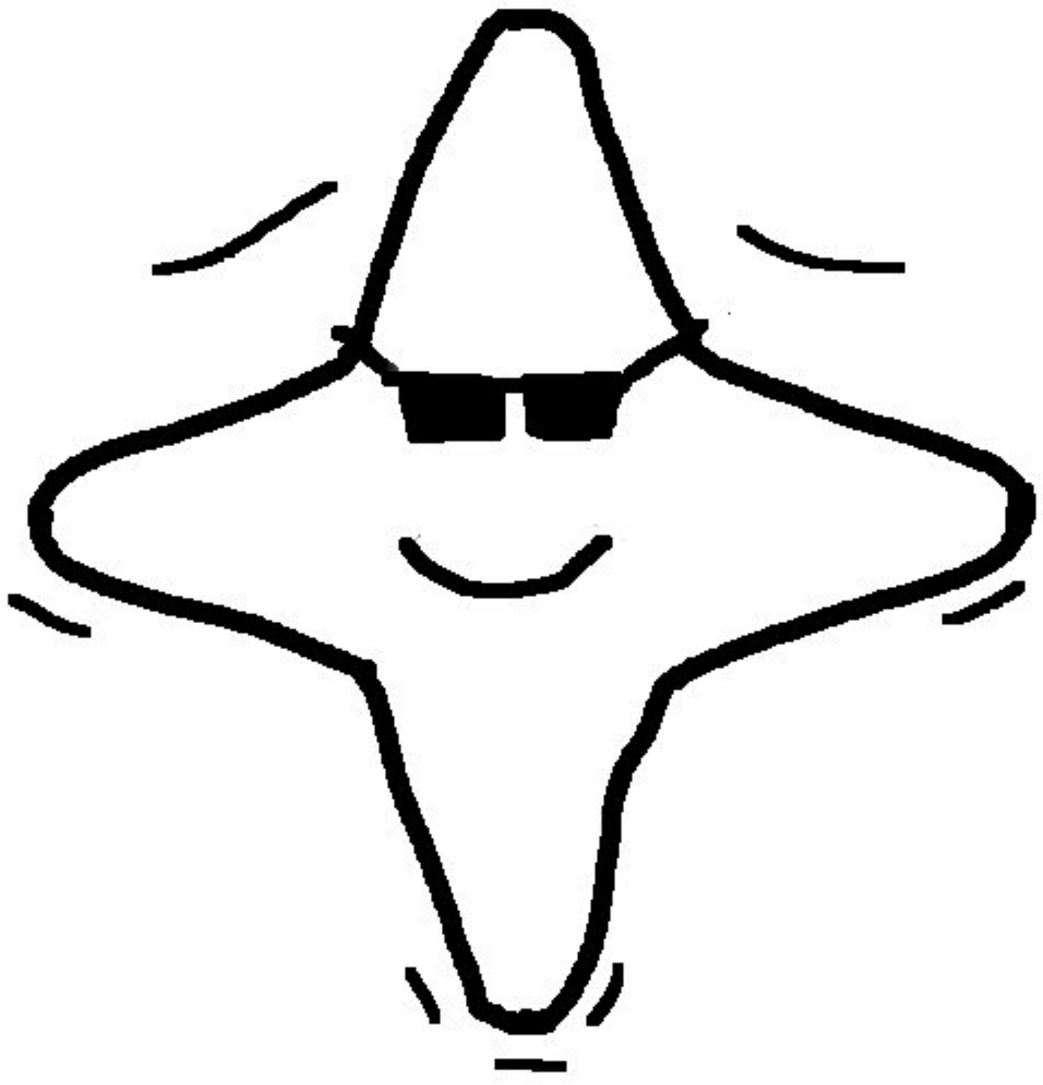

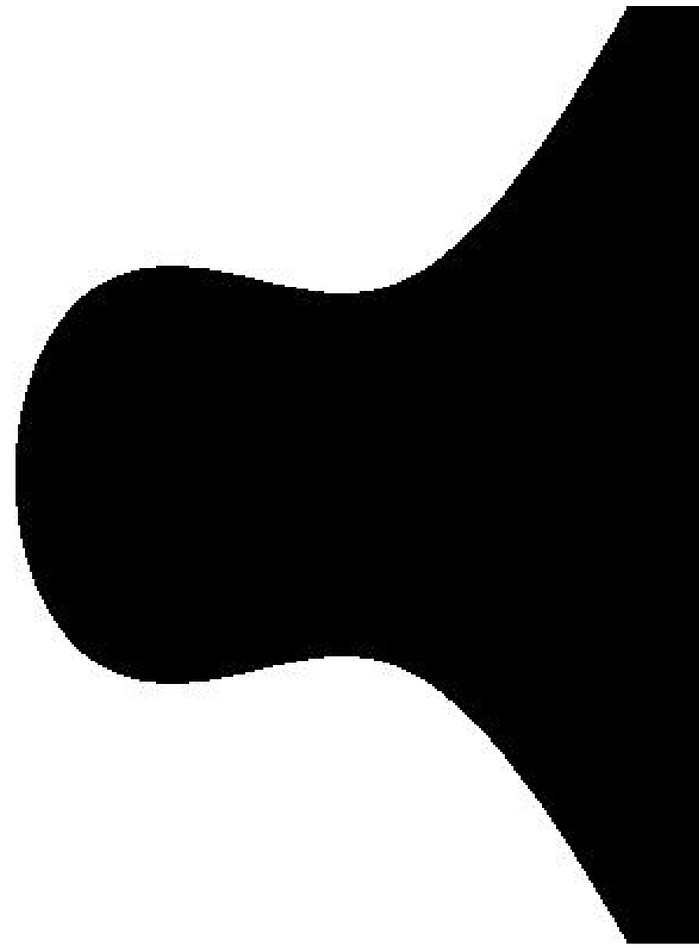

$$y^2 = x^3 - 0.4x + 0.7$$

2007 Bernstein–Lange:
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2008 Hisil–Wong–Carter–Dawson:
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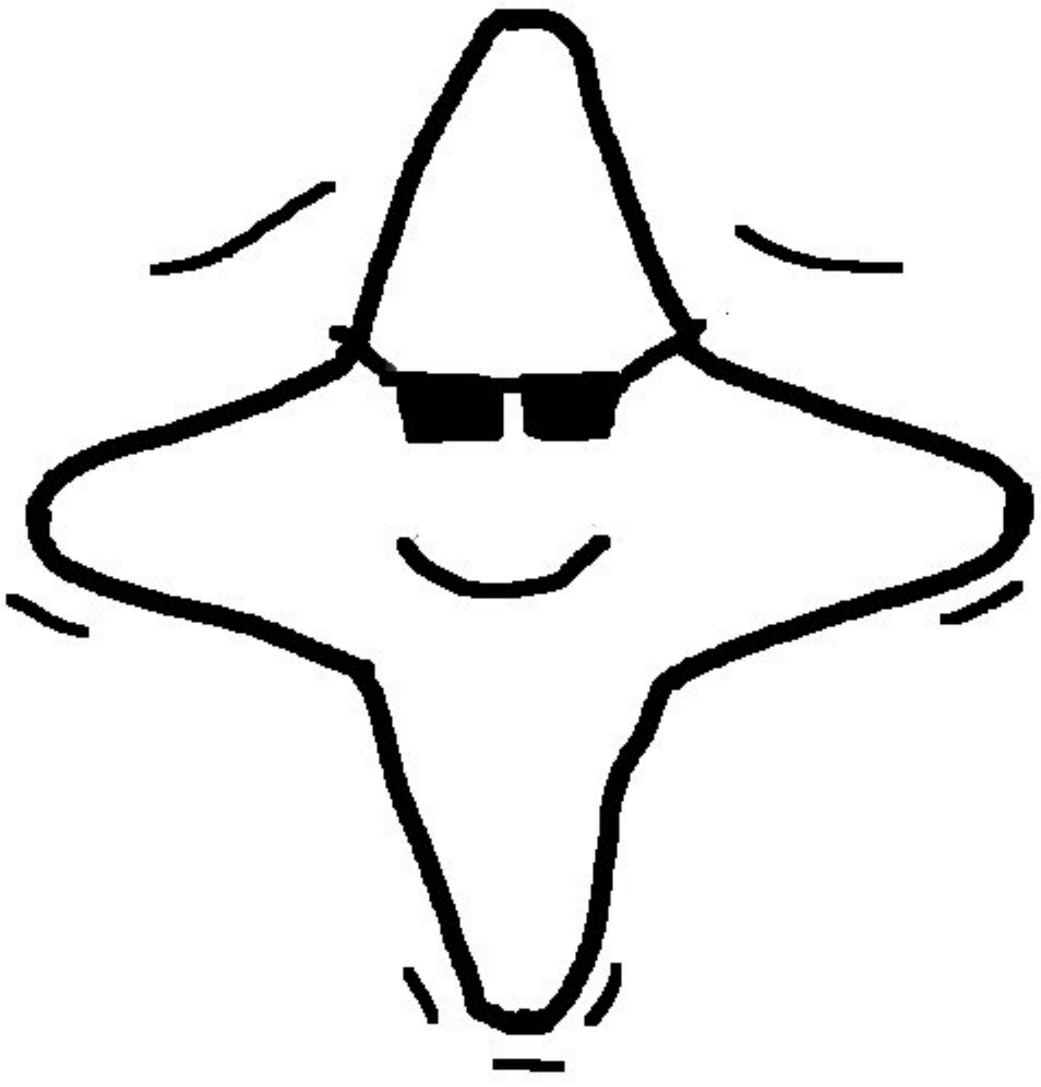
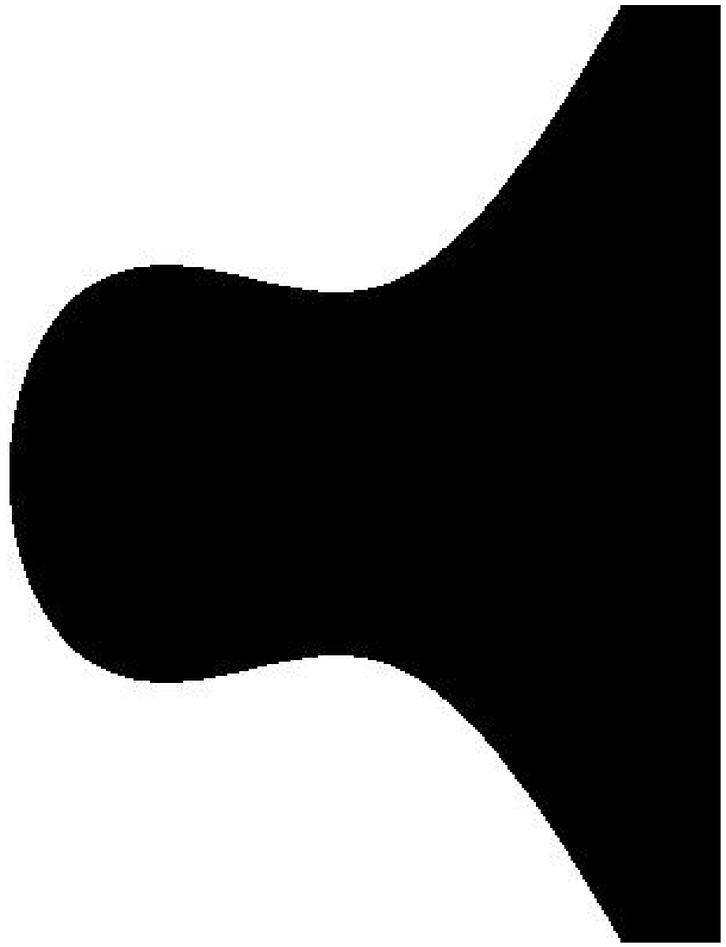




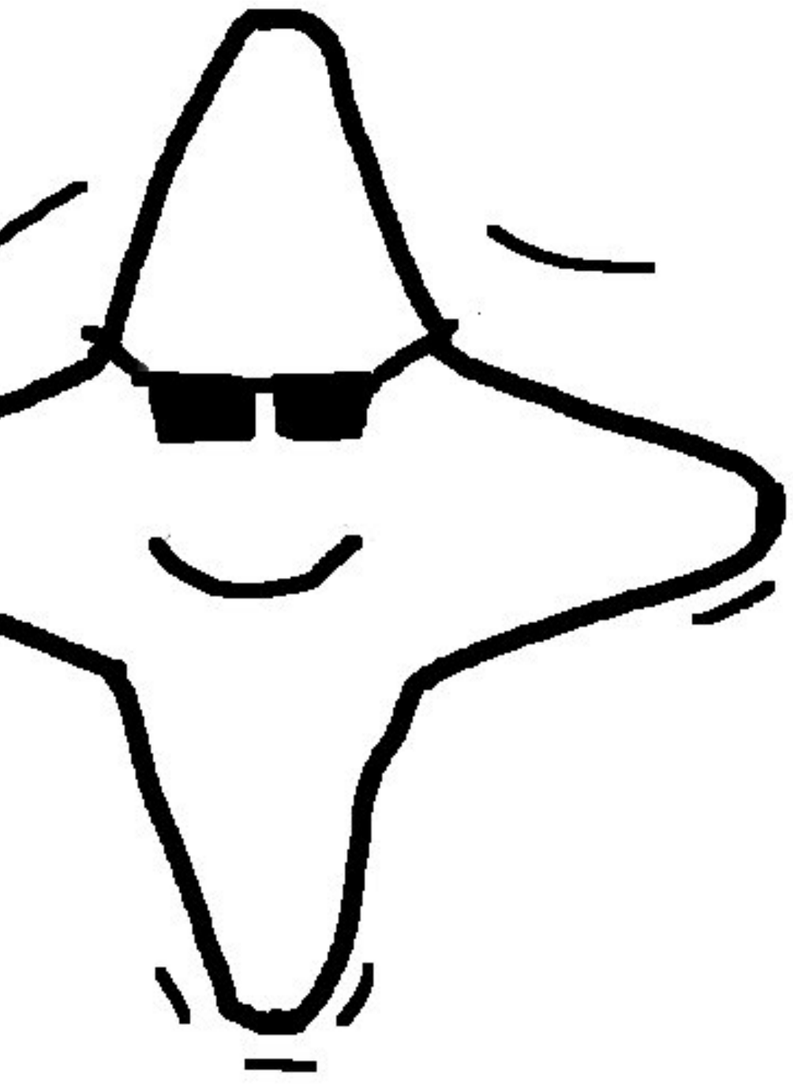
$y^2 = x^3 - 0.4x + 0.7$

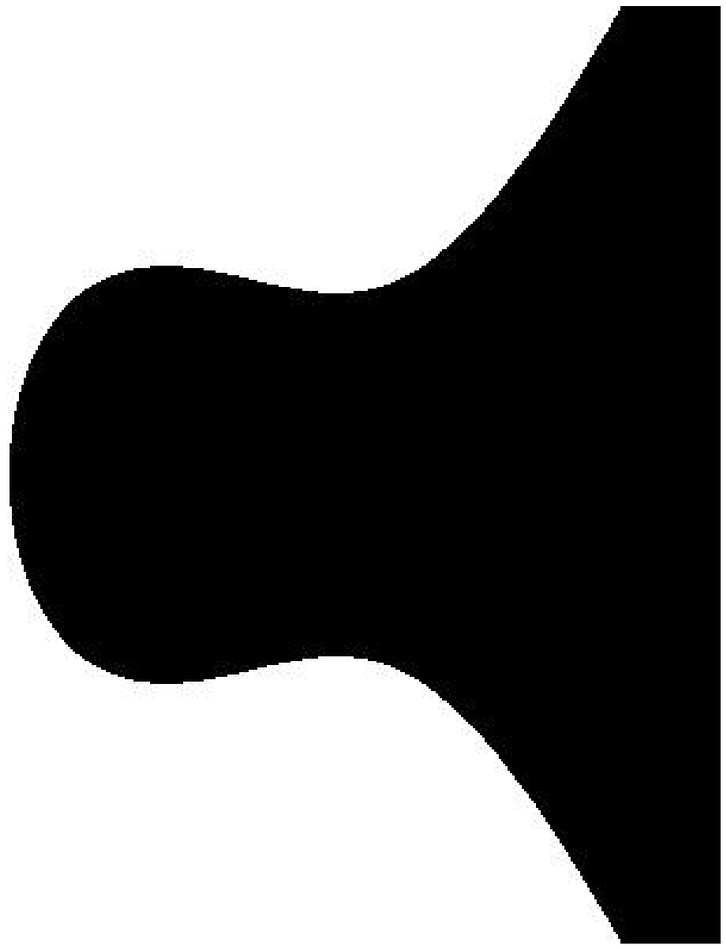
rnstein–Lange:

or ADD, 6.2**M** for DBL.

sil–Wong–Carter–Dawson:

for ADD.





$$y^2 = x^3 - 0.4x + 0.7$$

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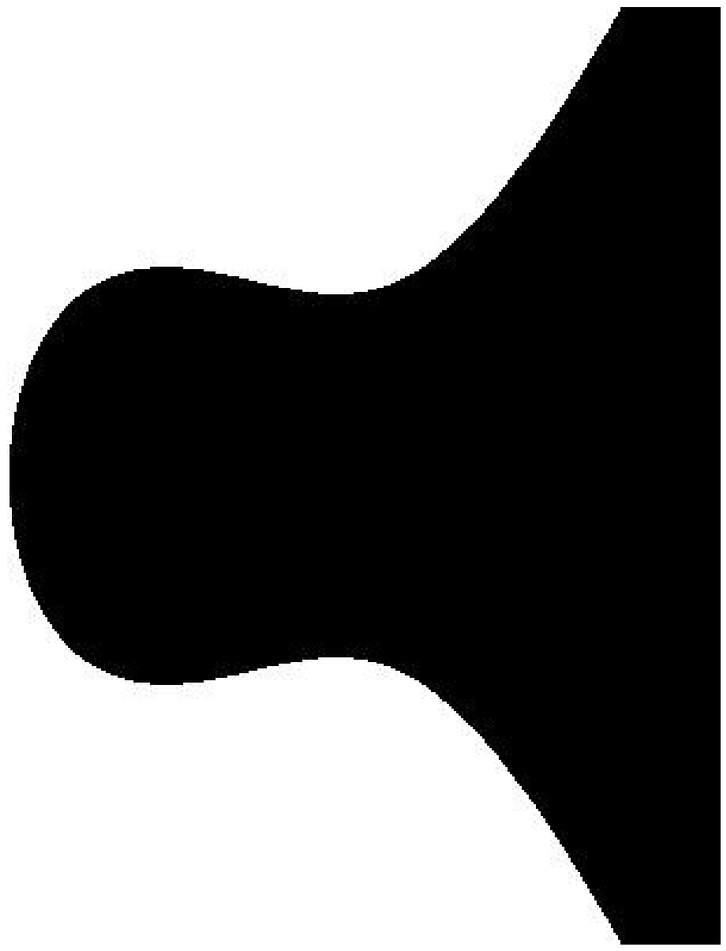
6.2**M** for DBL.

Carter–Dawson:



$$y^2 = x^3 - 0.4x + 0.7$$

The Weierstrass-
turtle: old, trusted
and slow. Warning
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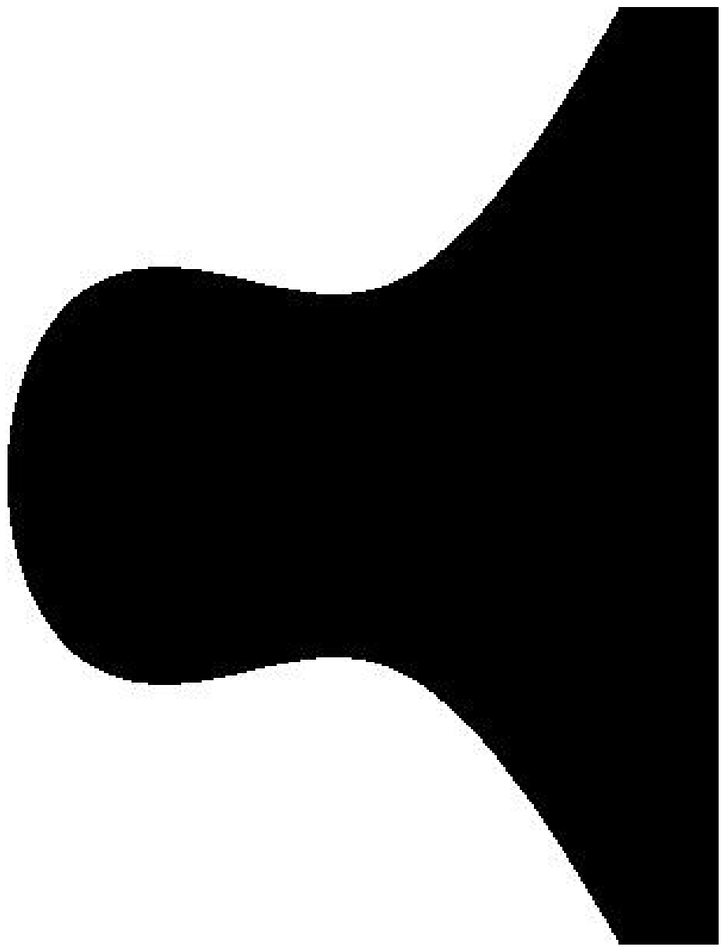
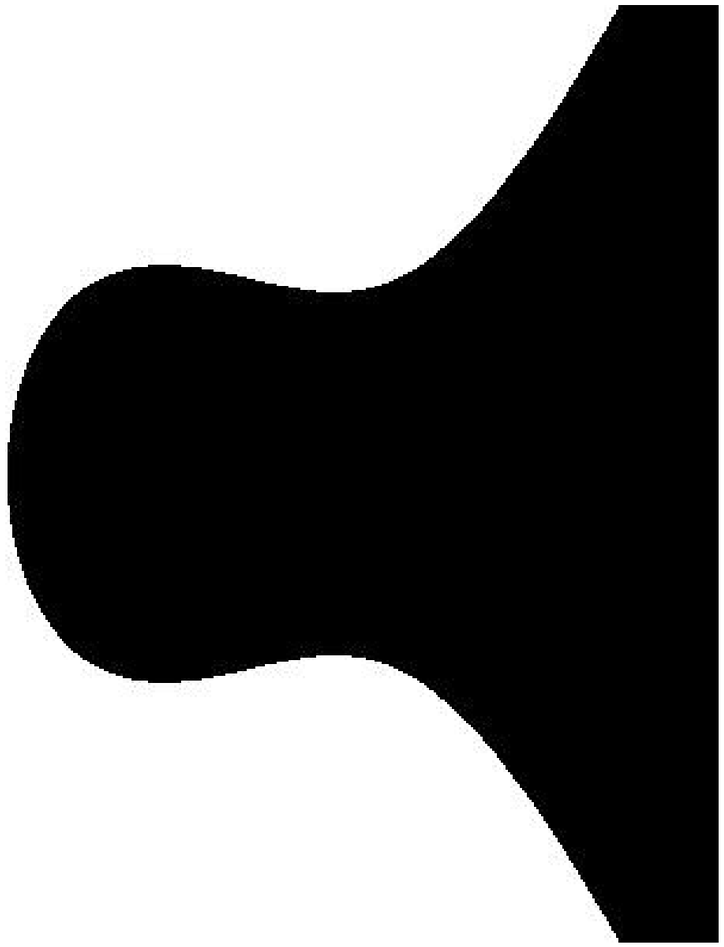


$$y^2 = x^3 - 0.4x + 0.7$$



$$y^2 = x^3 - 0.4x + 0.7$$


The Weierstrass-
turtle: old, trusted
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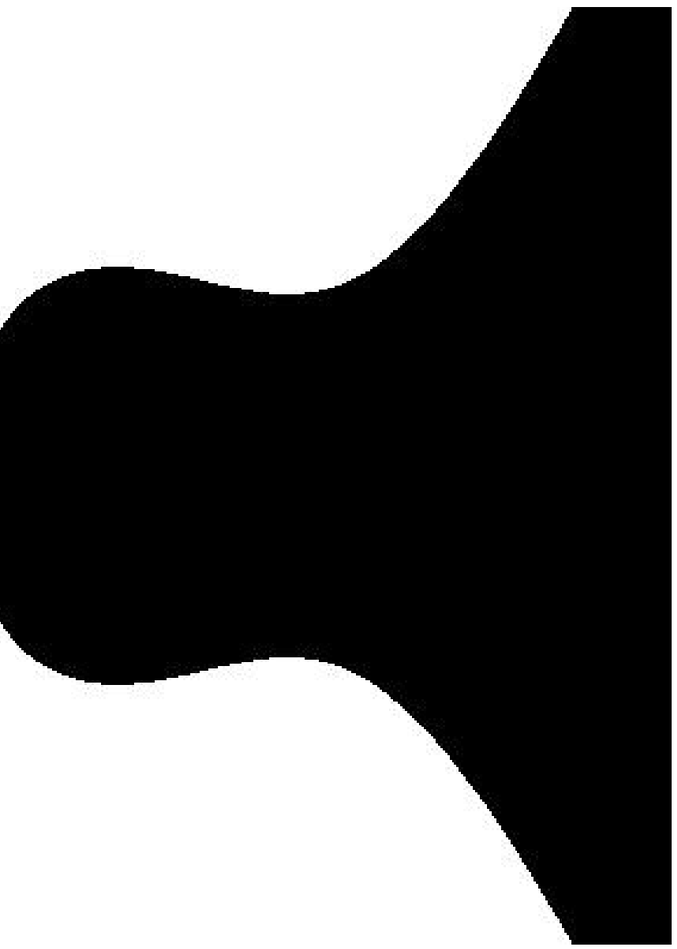

$$-0.4x + 0.7$$


The Weierstrass-
turtle: old, trusted
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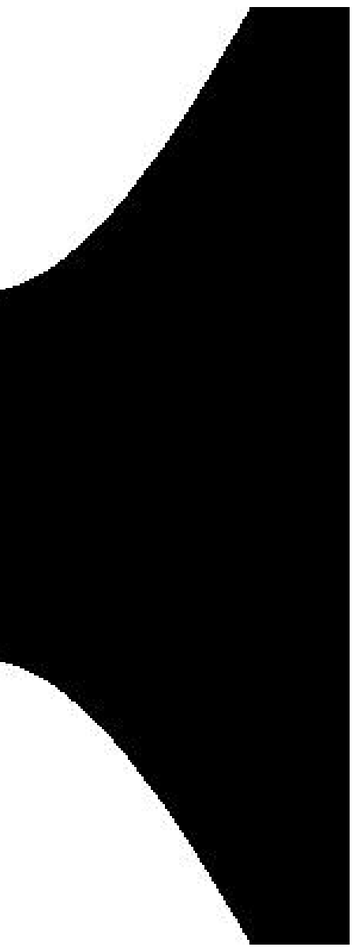
$$x^2 + y^2$$



$$\frac{}{0.7}$$

The Weierstrass-
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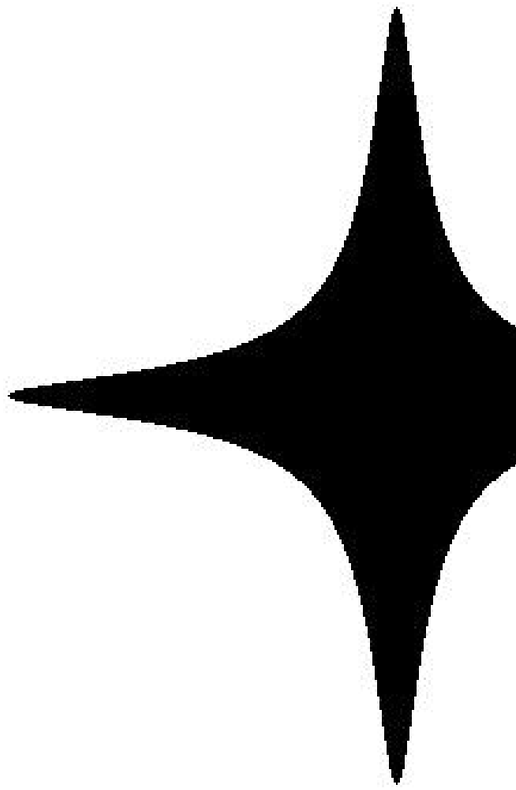
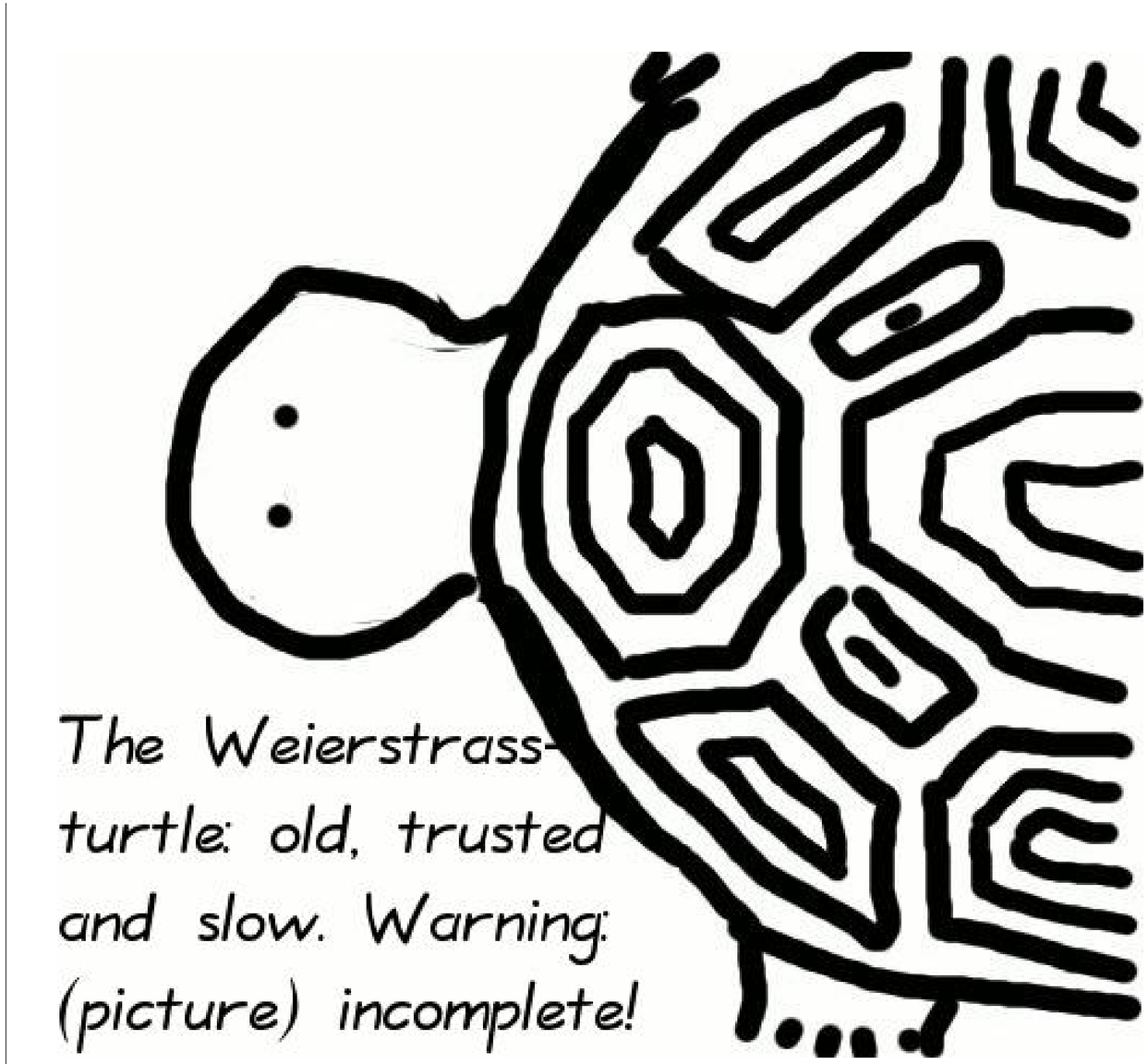
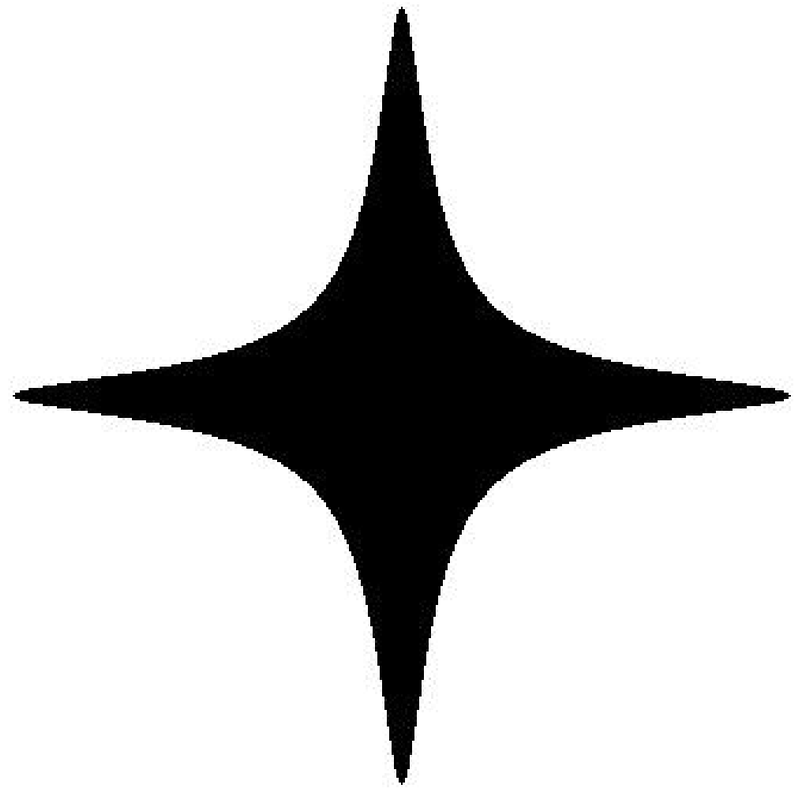
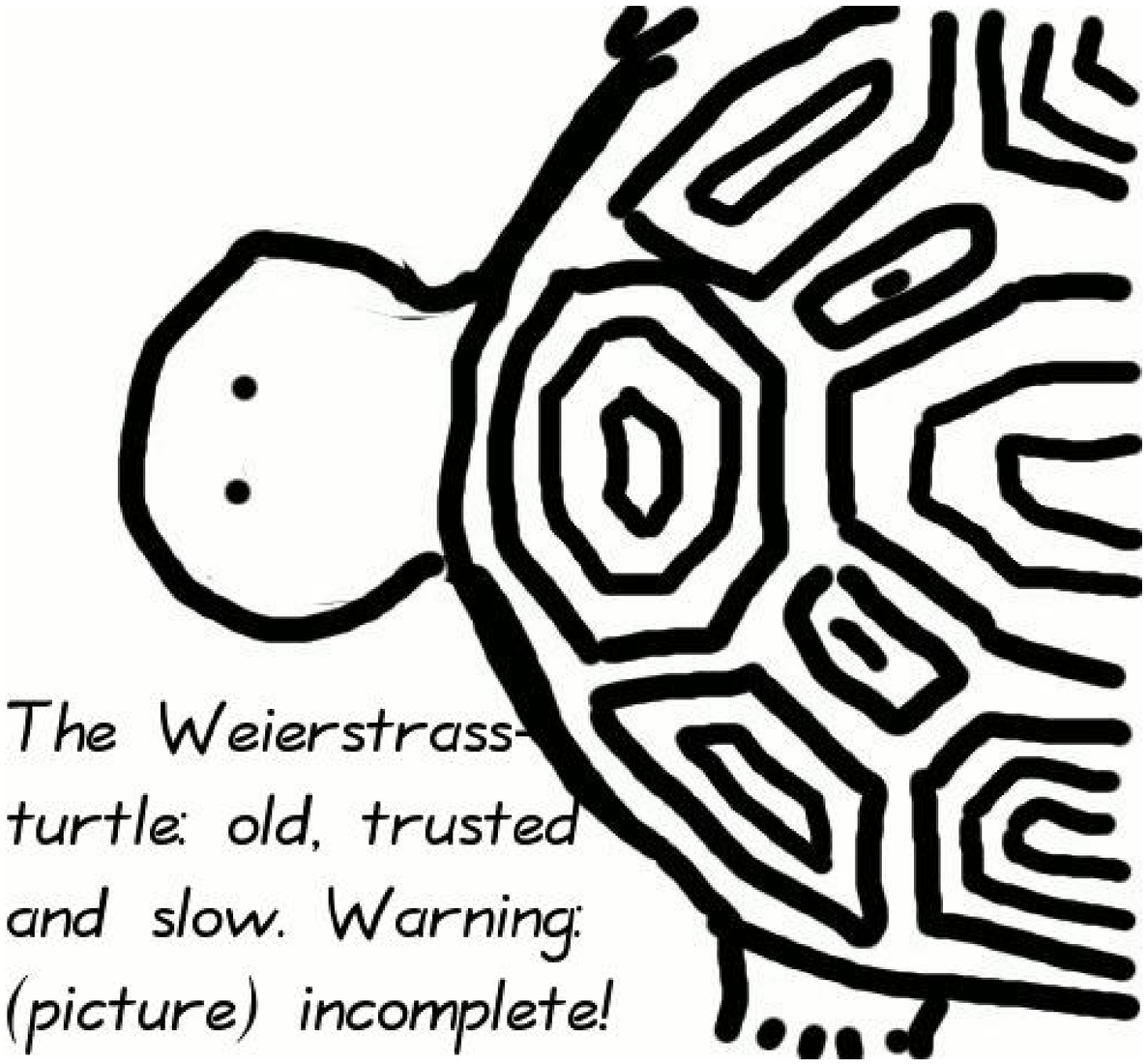


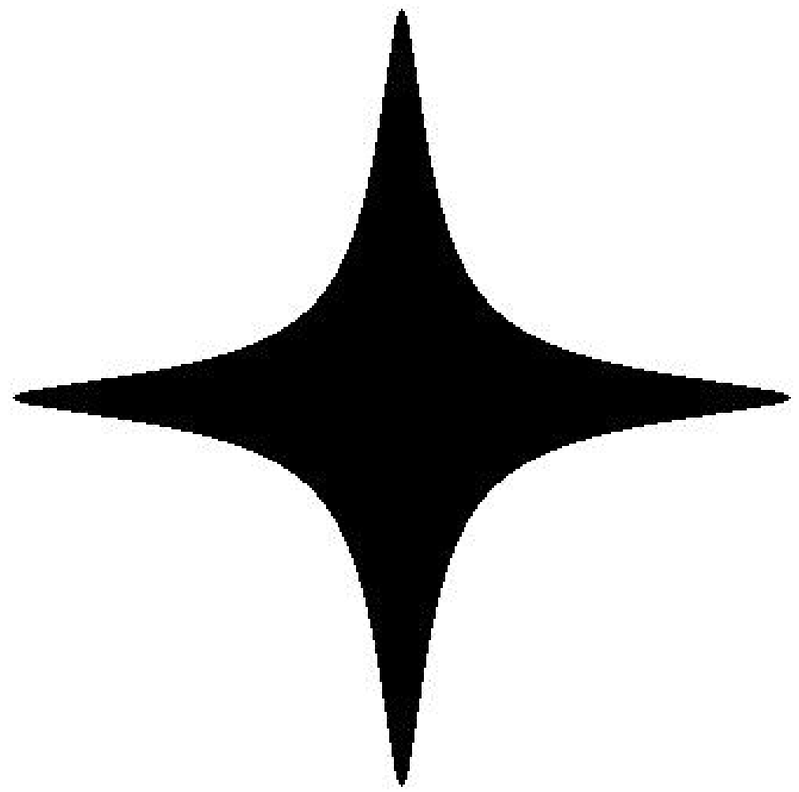
$$x^2 + y^2 = 1 - 300$$





$$x^2 + y^2 = 1 - 300x^2y^2$$





$$x^2 + y^2 = 1 - 300x^2y^2$$

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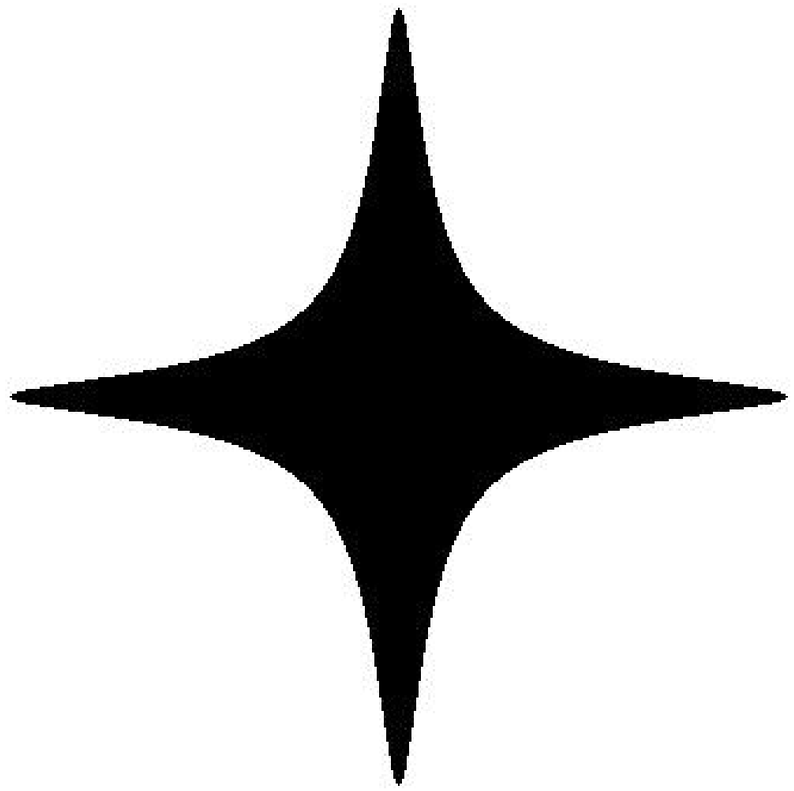

$$x^2 + y^2 = 1 - 300x^2y^2$$

The Edv
starfish
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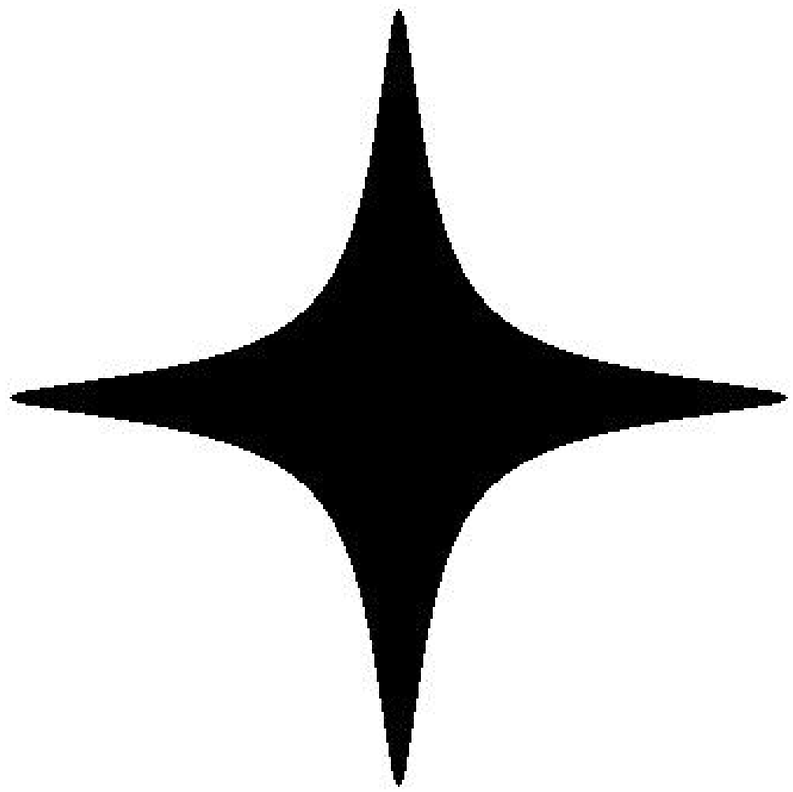
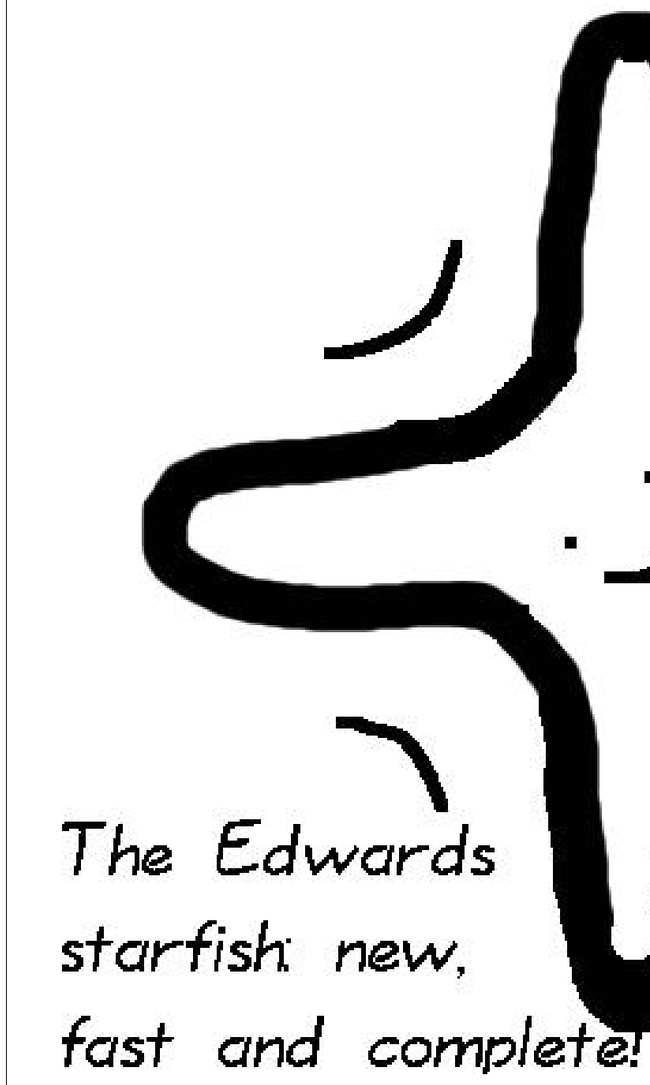
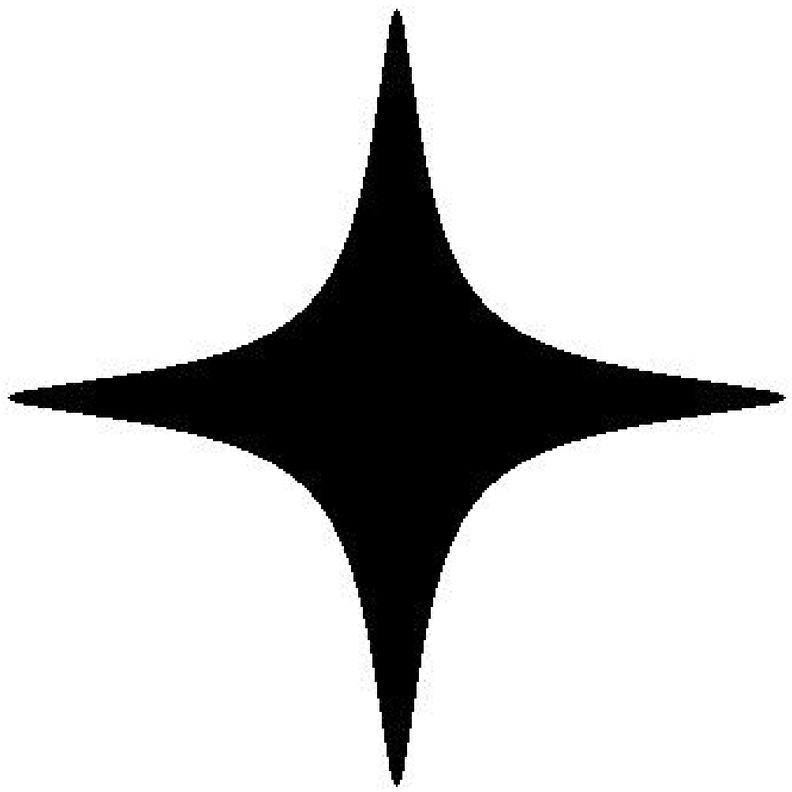
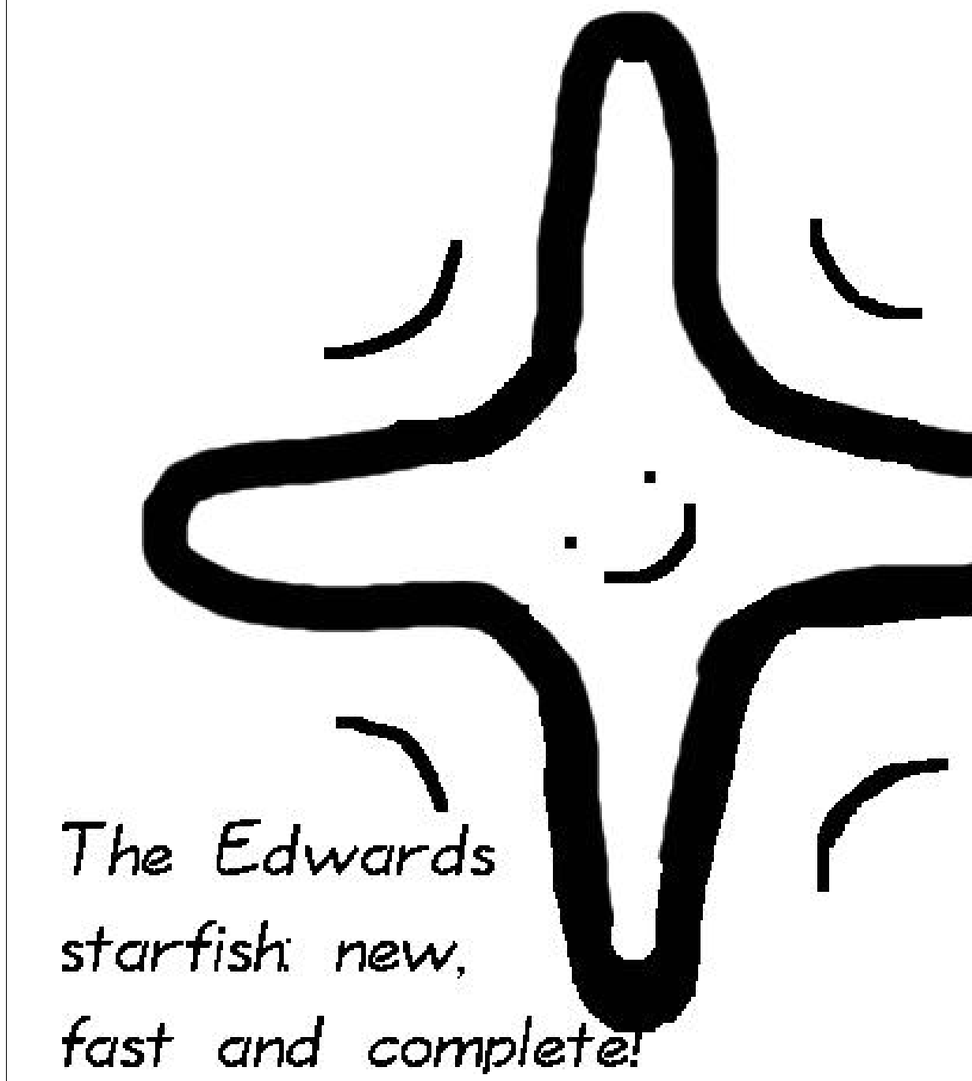


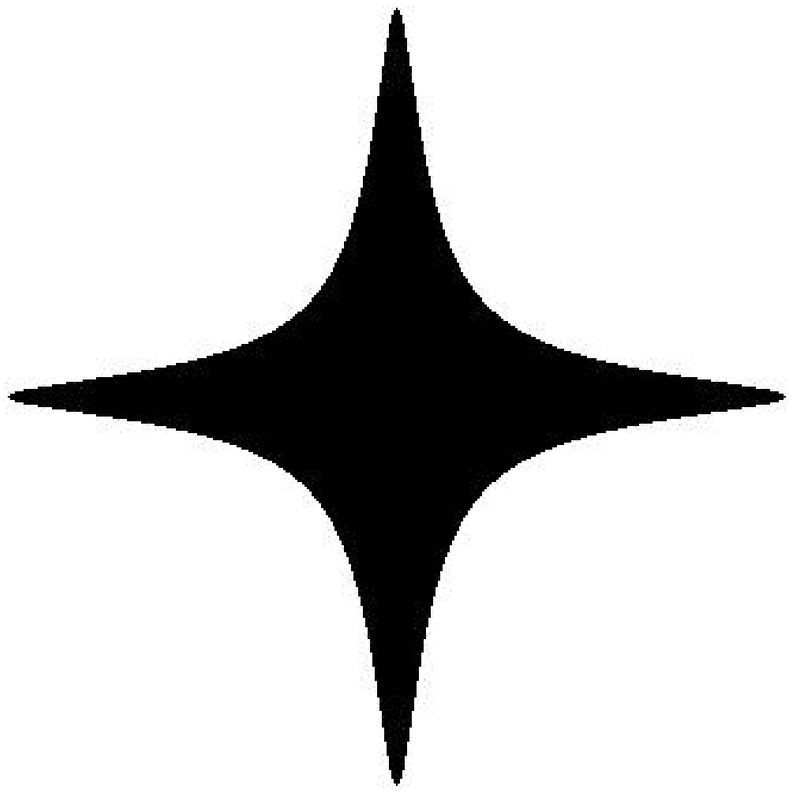
$$x^2 + y^2 = 1 - 300x^2y^2$$



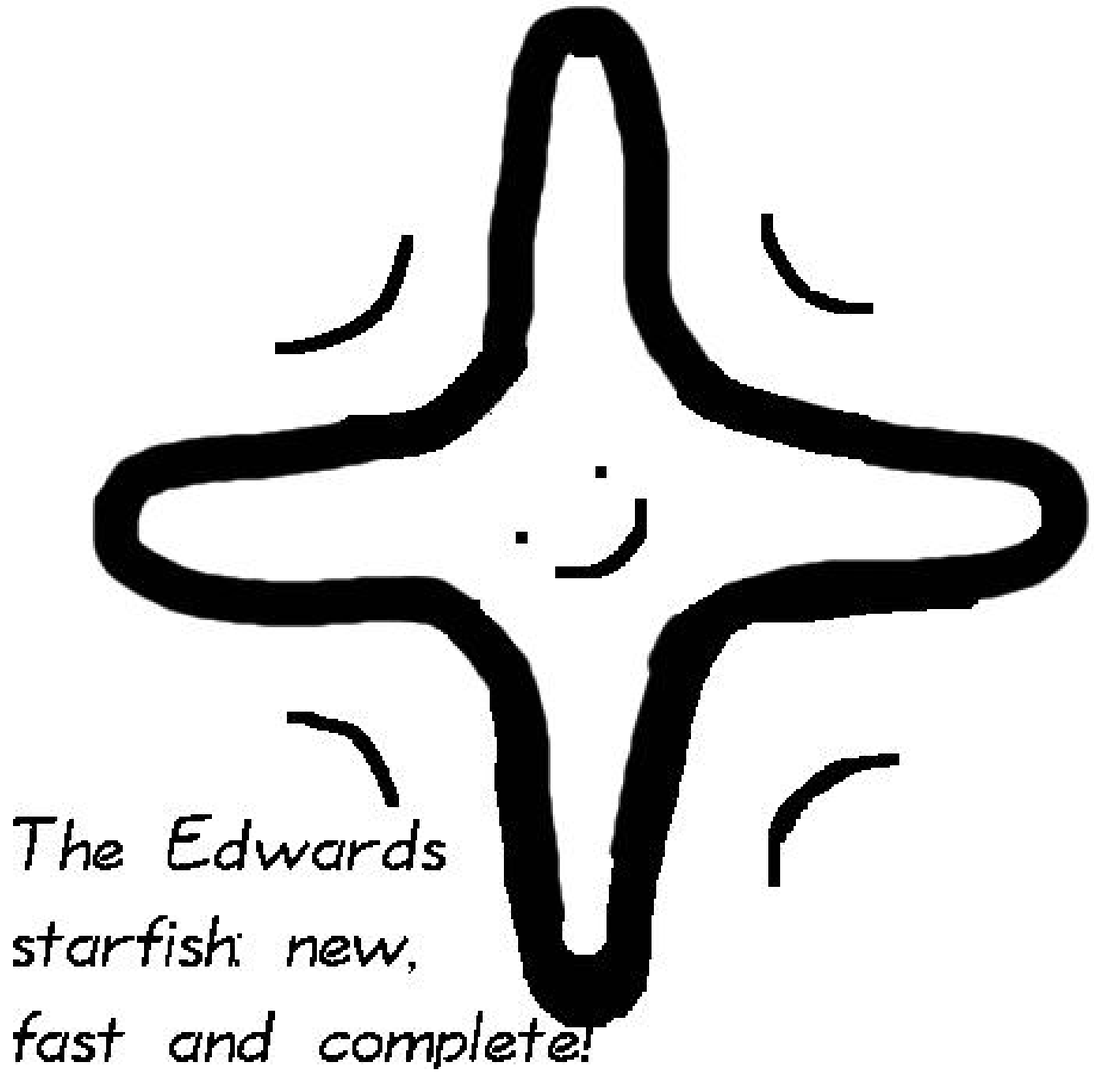


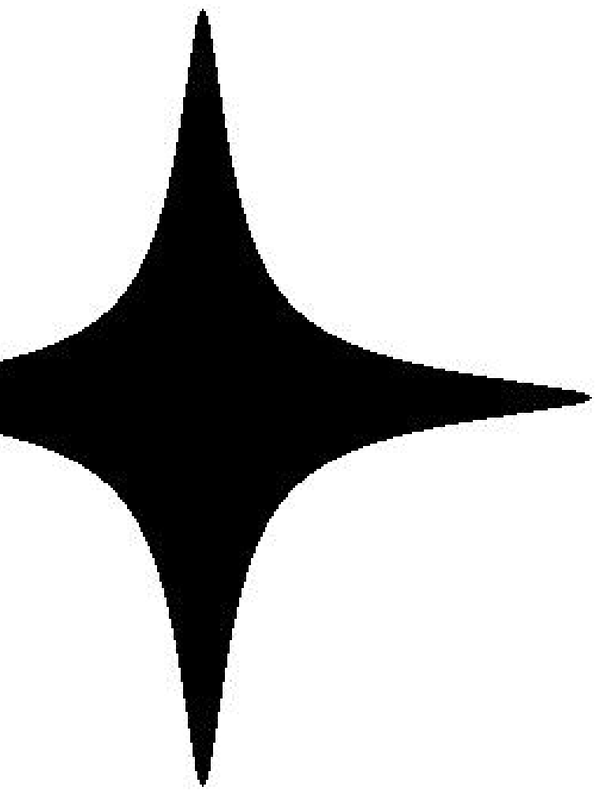
$$x^2 + y^2 = 1 - 300x^2y^2$$



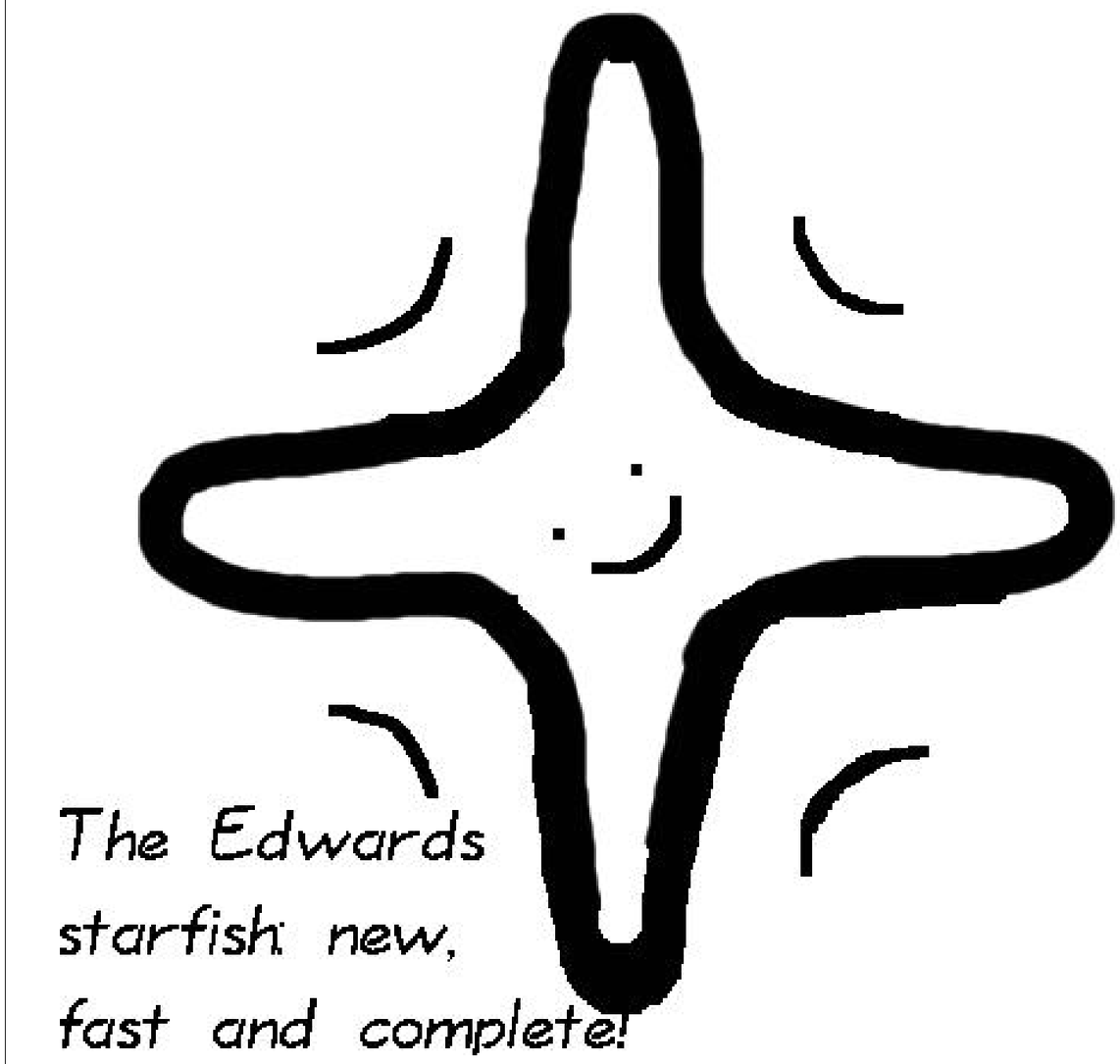


$$x^2 + y^2 = 1 - 300x^2y^2$$

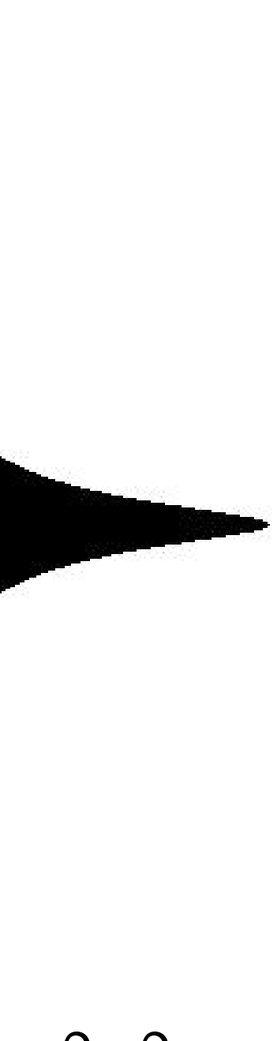




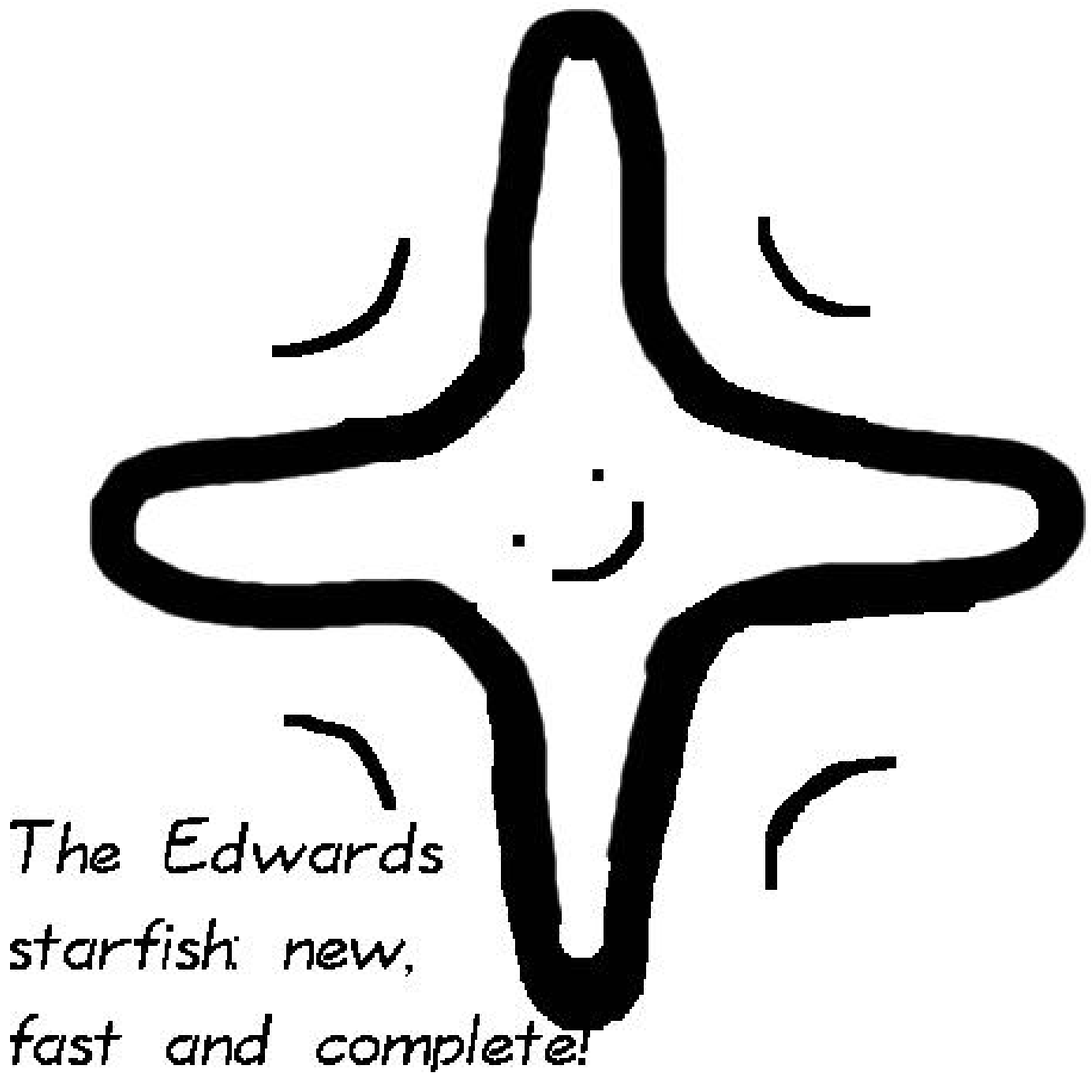
$= 1 - 300x^2y^2$

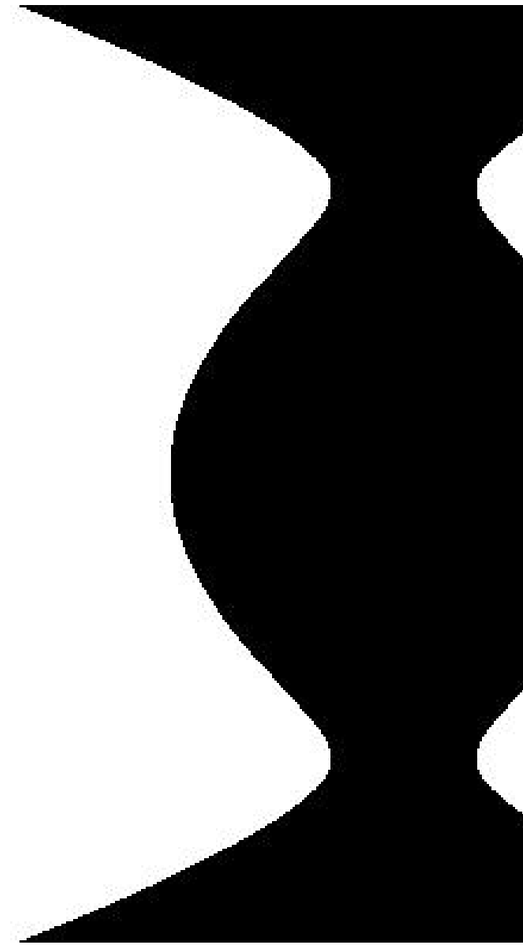


$x^2 = y^4$

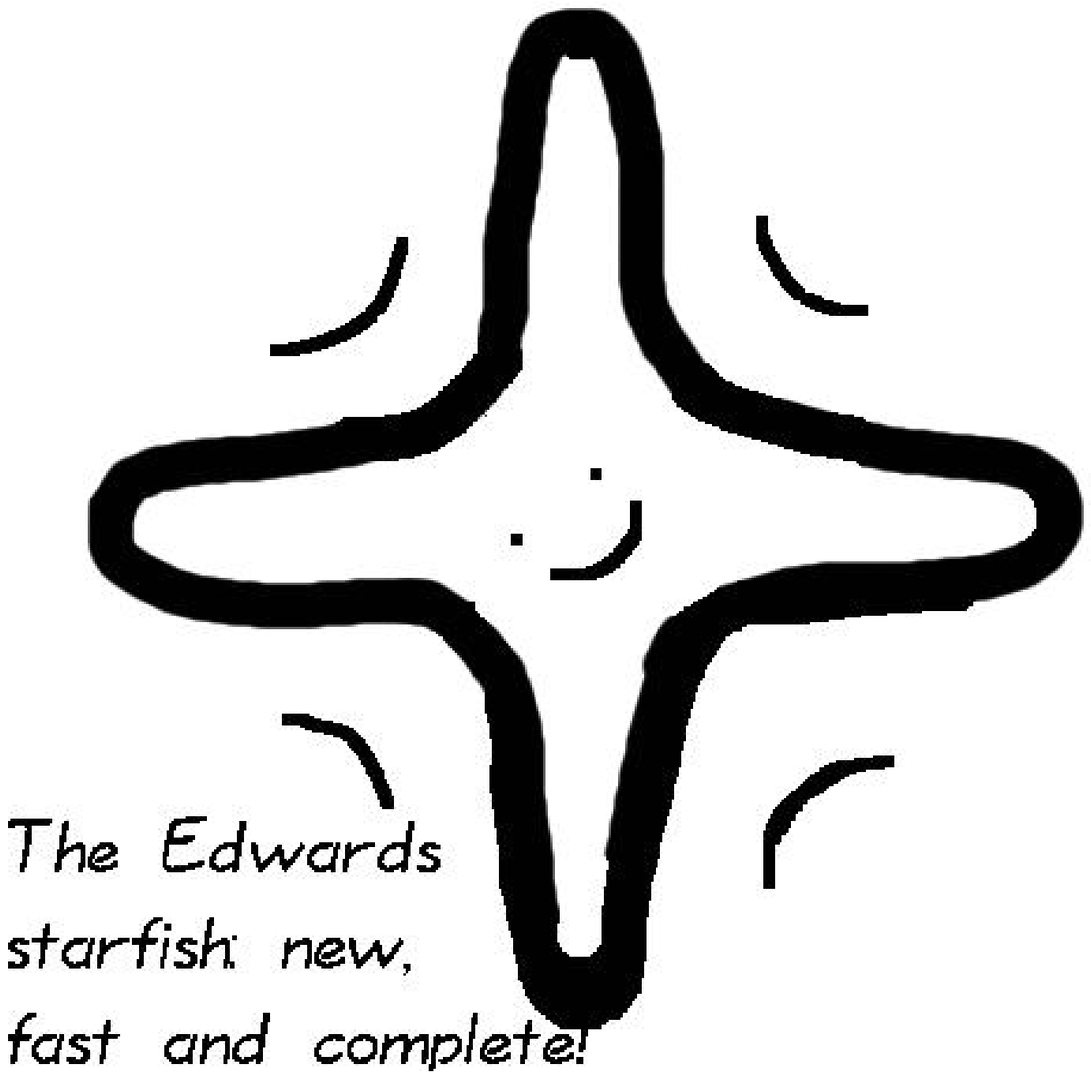


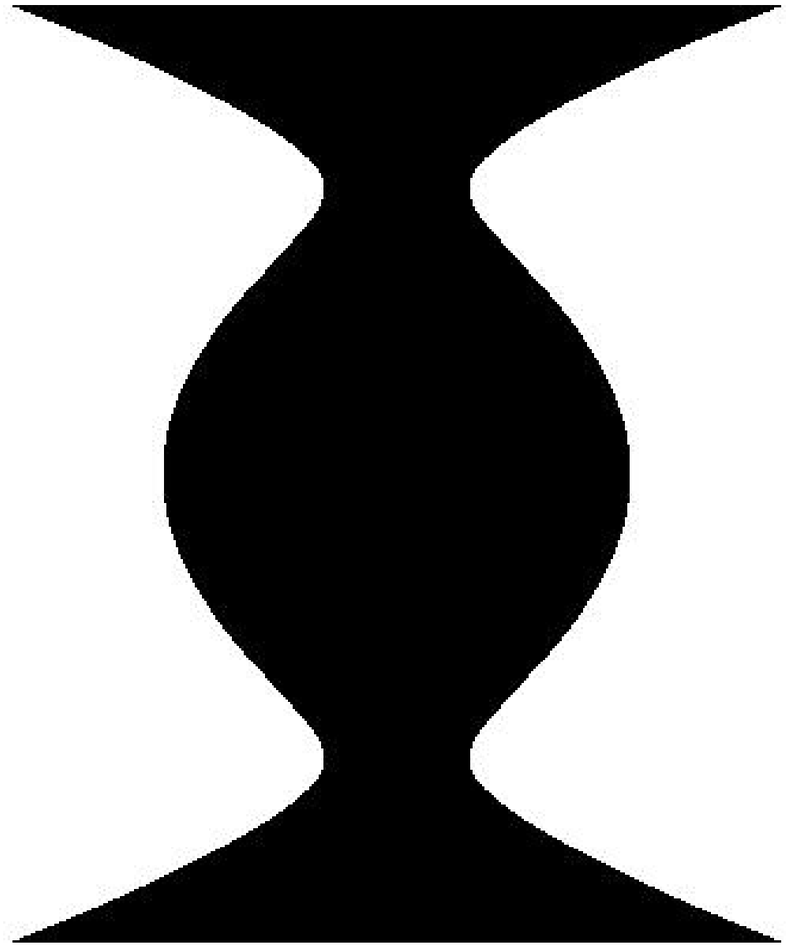
$0x^2y^2$



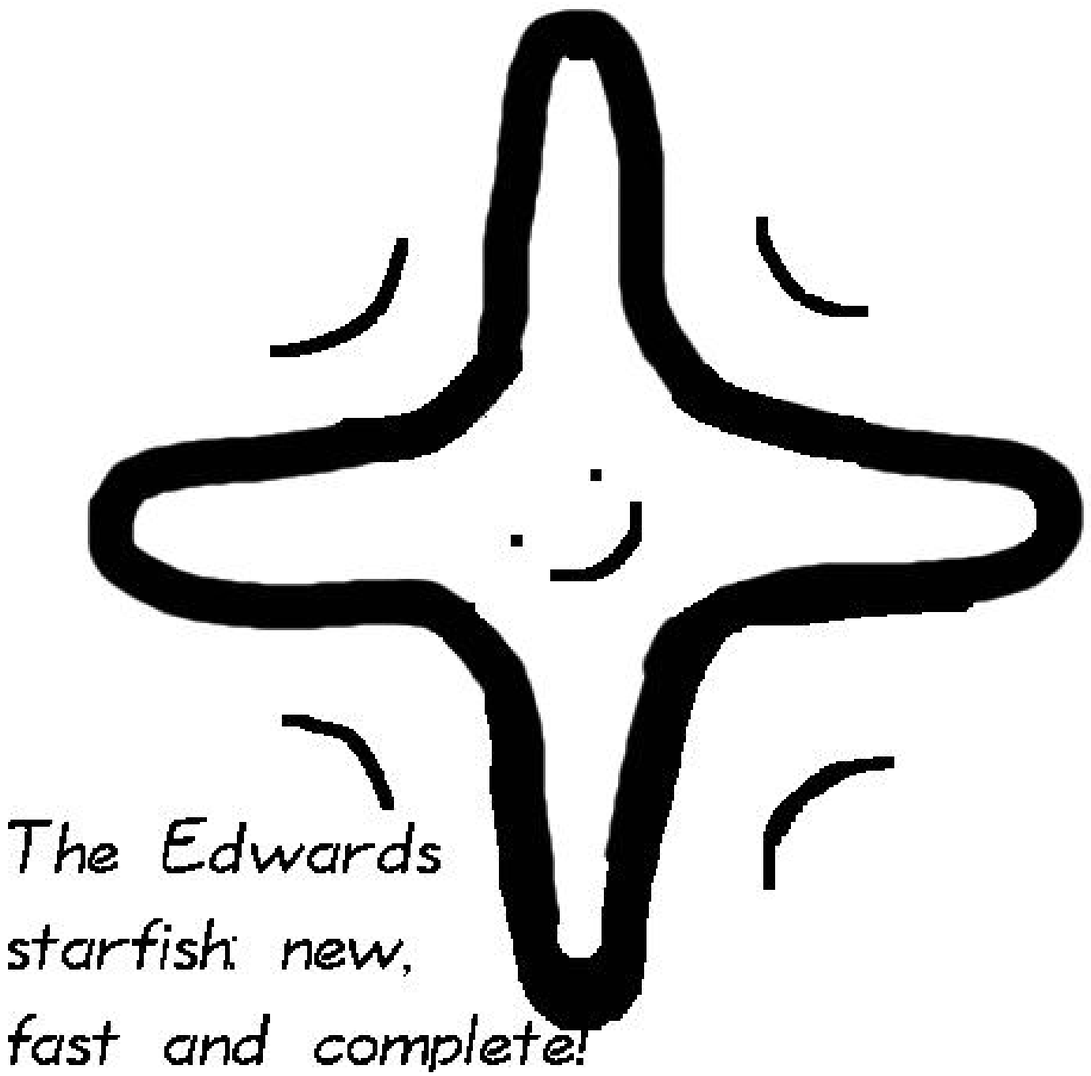


$x^2 = y^4 - 1.9y^2 +$

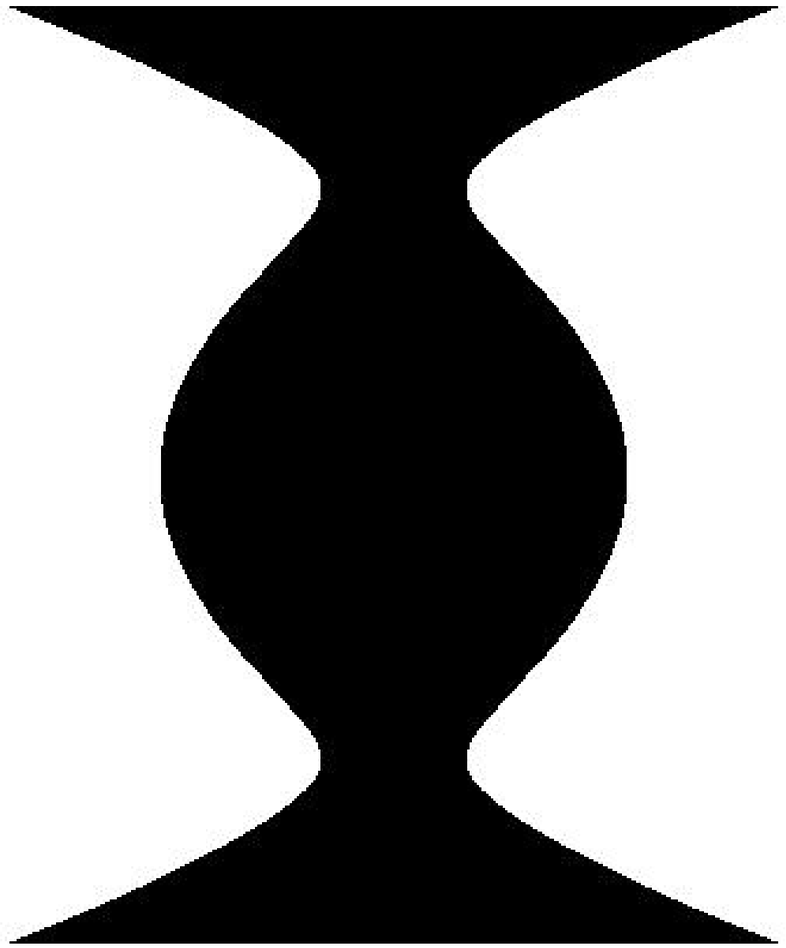




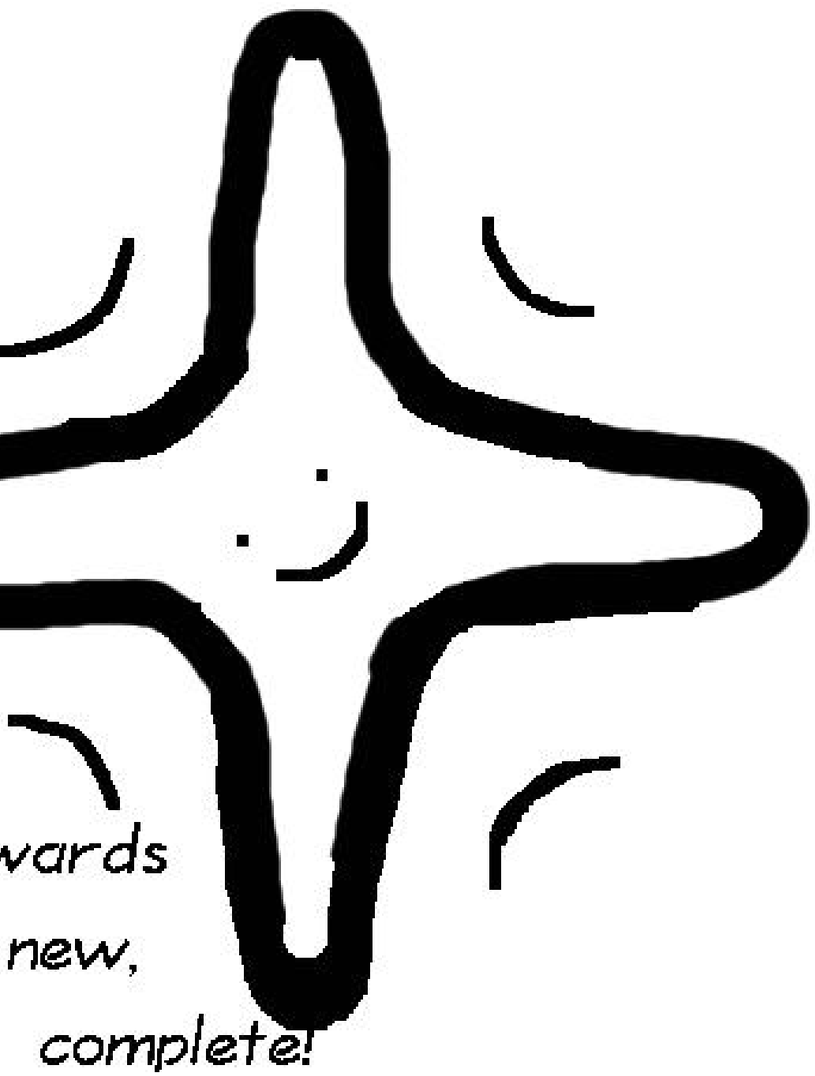
$$x^2 = y^4 - 1.9y^2 + 1$$

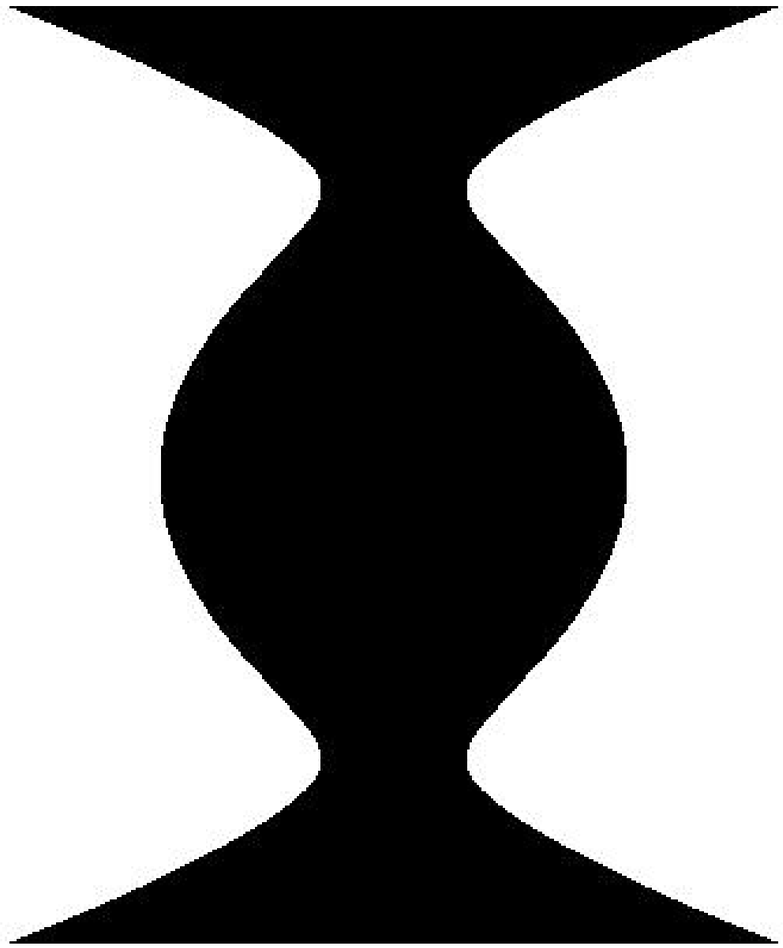


The Edwards starfish: new, fast and complete!

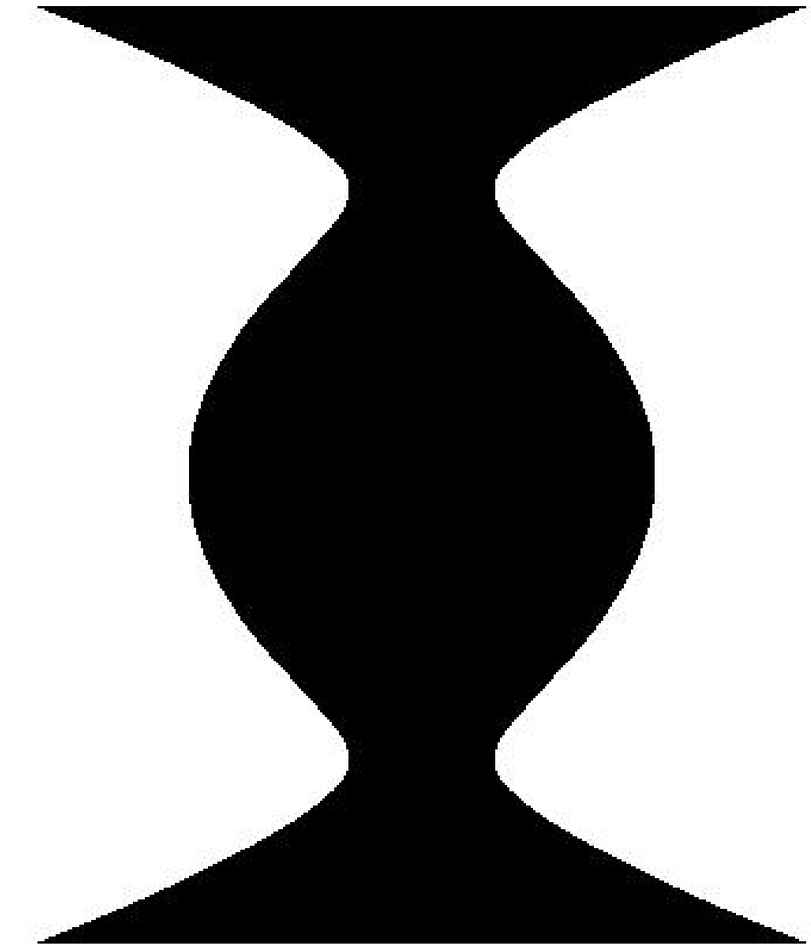


$$x^2 = y^4 - 1.9y^2 + 1$$



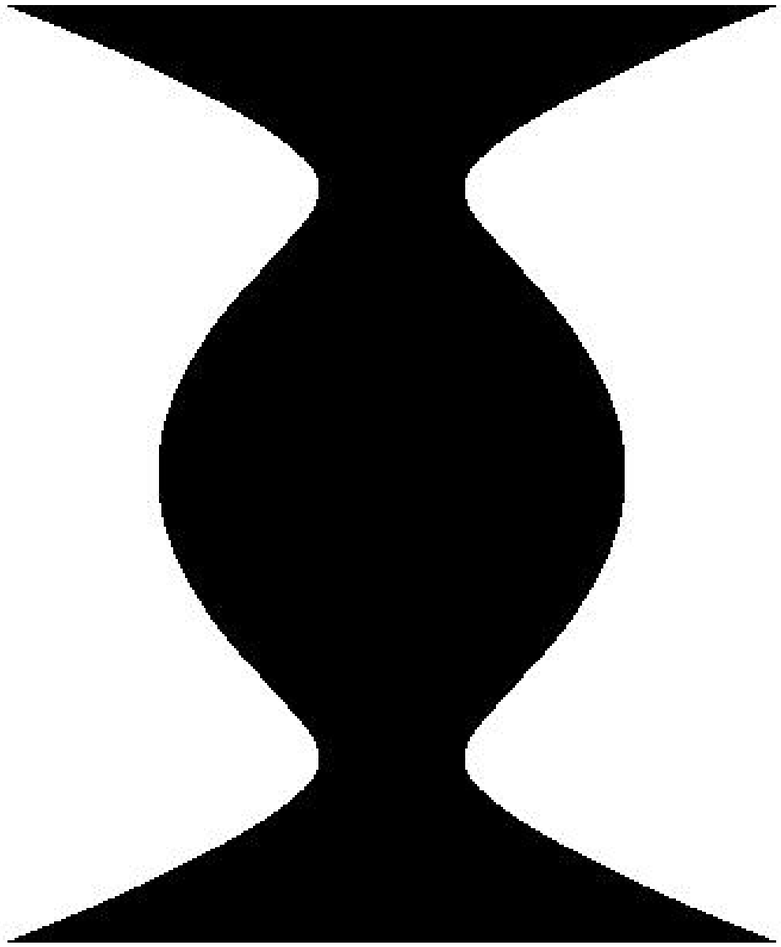


$$x^2 = y^4 - 1.9y^2 + 1$$



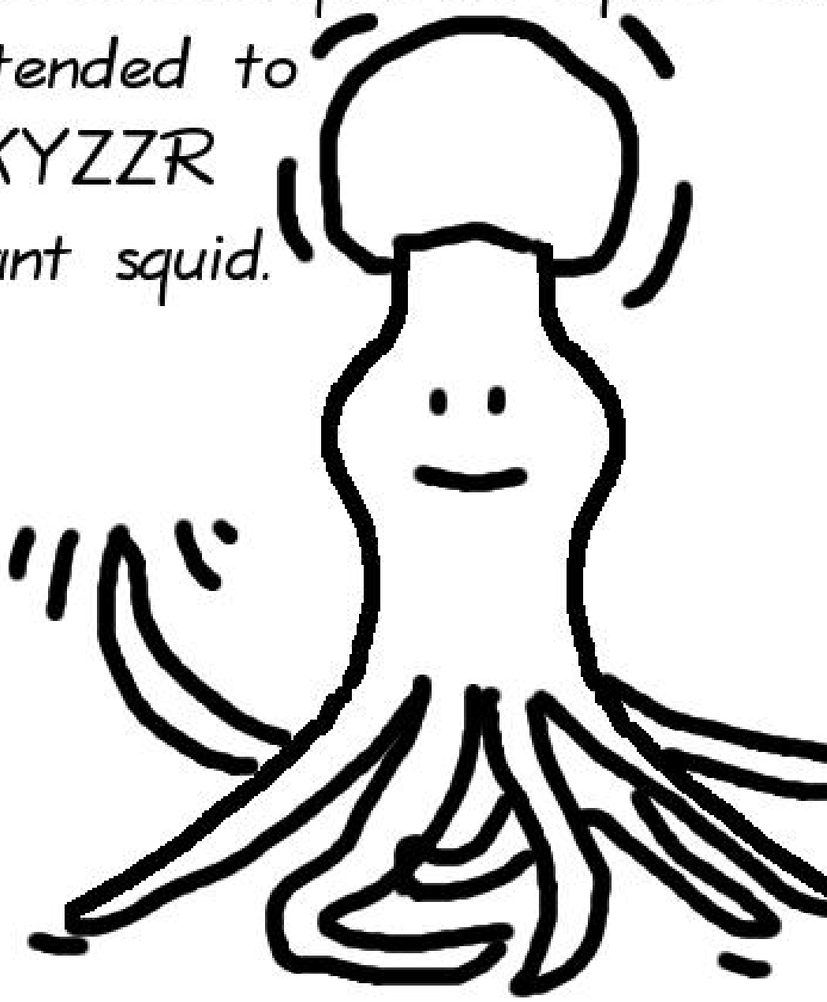
$$x^2 = y^4 - 1.9y^2 + 1$$

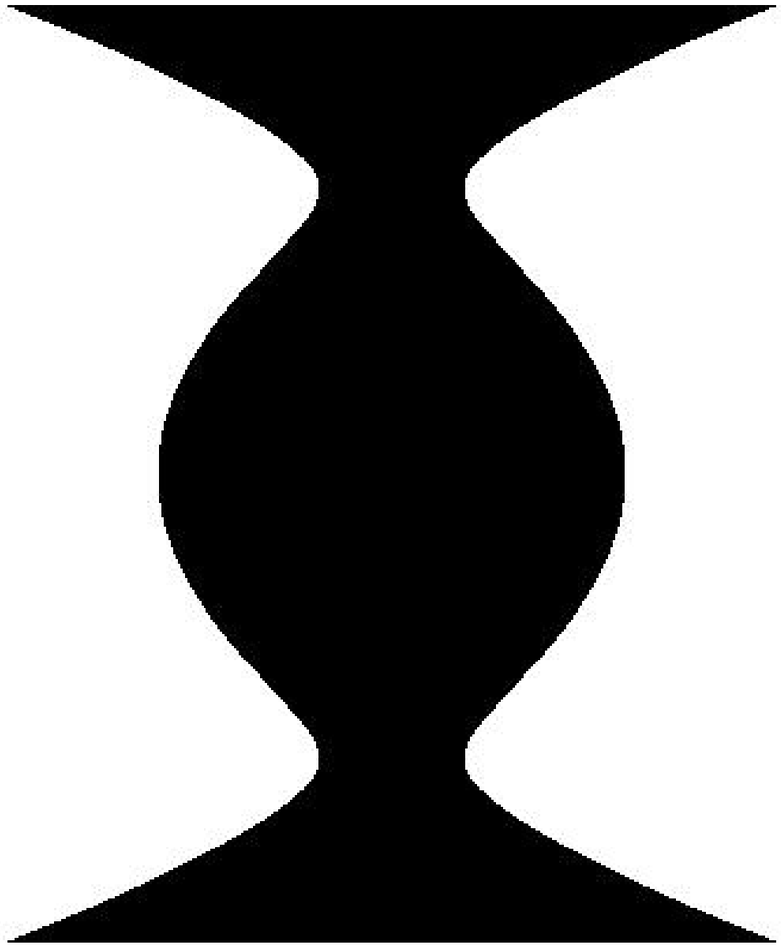
The Jacobi-quartic extended to XXYZZR giant squid.



$$x^2 = y^4 - 1.9y^2 + 1$$

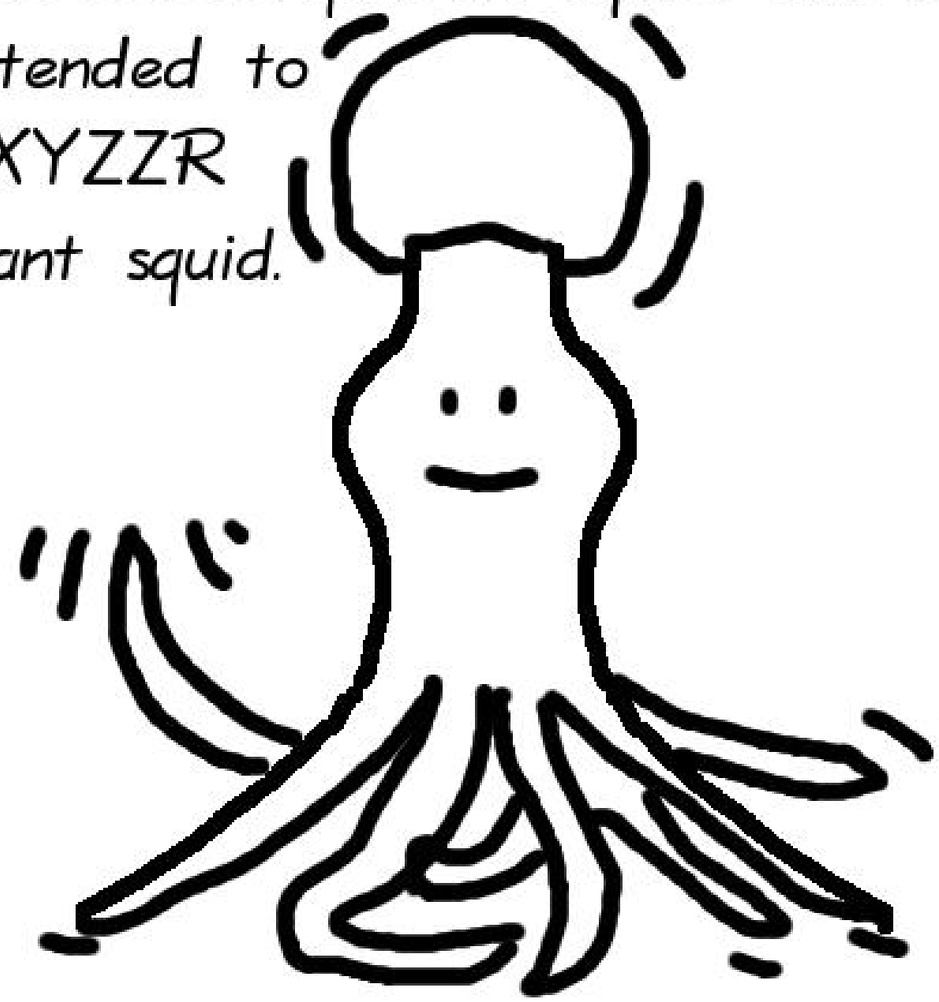
The Jacobi-quartic squid: ca
extended to
XXYZZR
giant squid.

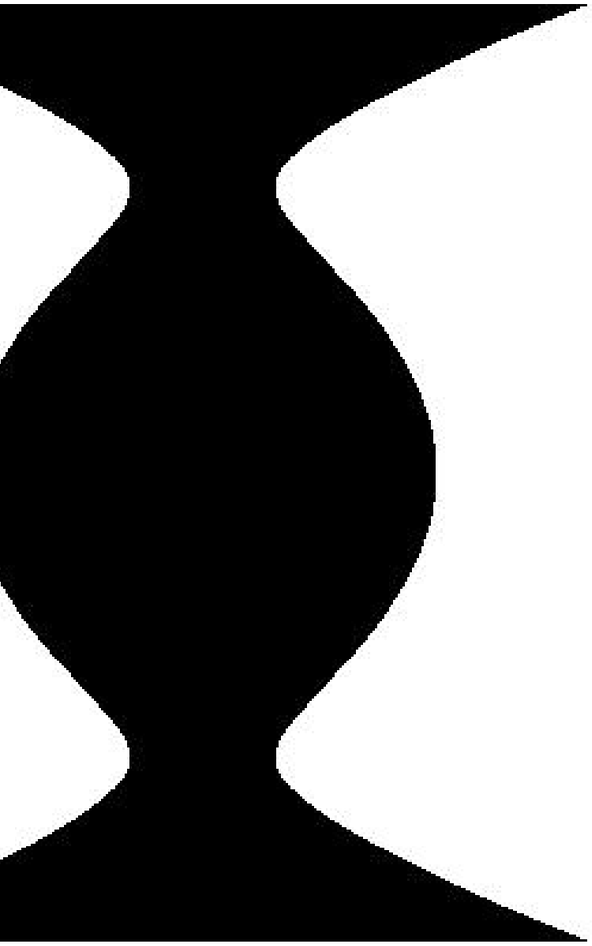




$x^2 = y^4 - 1.9y^2 + 1$

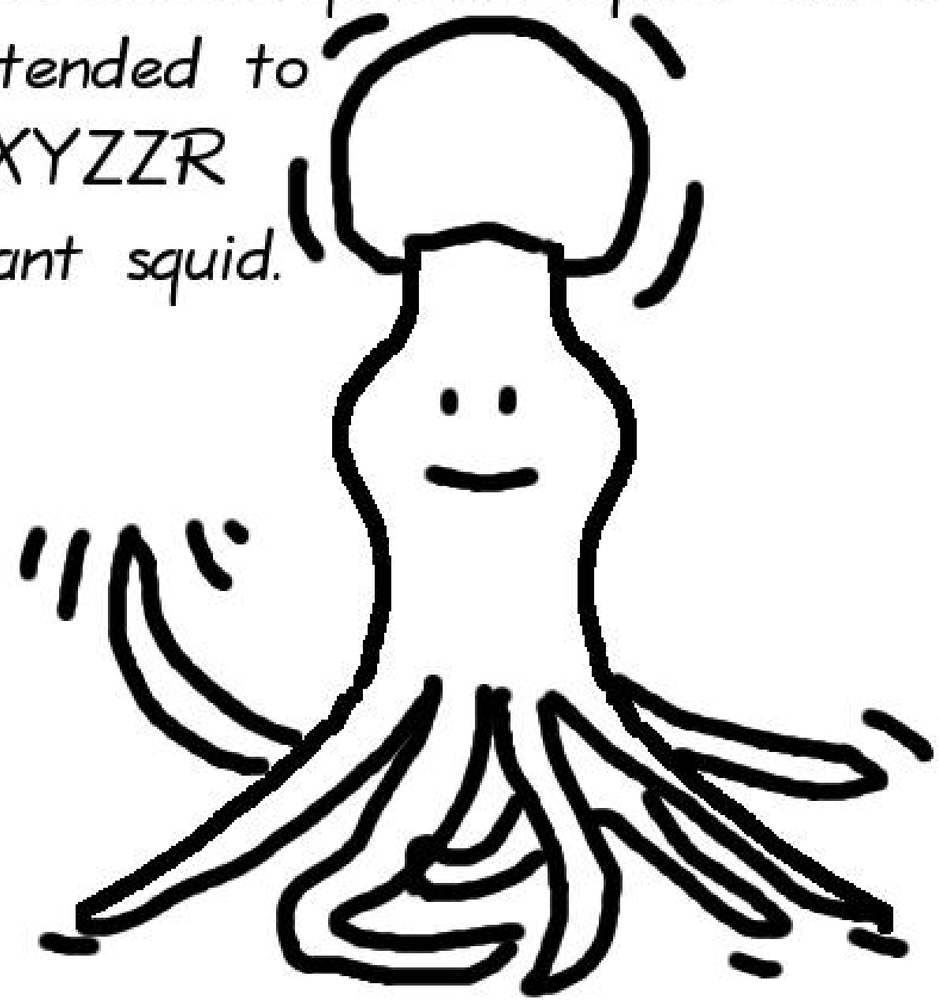
The Jacobi-quartic squid: can be extended to XXYZZR giant squid.

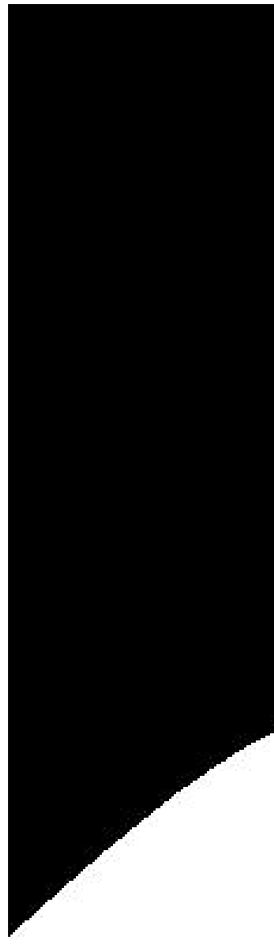




$-1.9y^2+1$

The Jacobi-quartic squid: can be extended to XXYZZR giant squid.

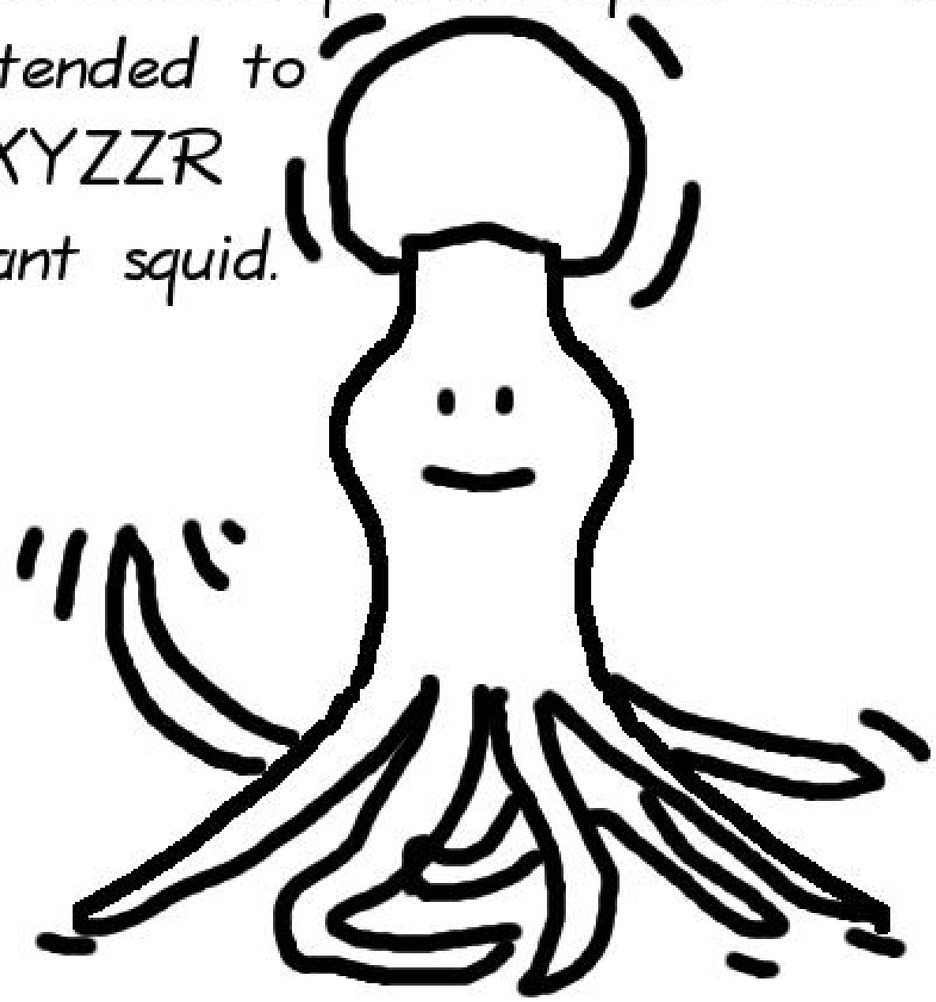




x^3-y^3

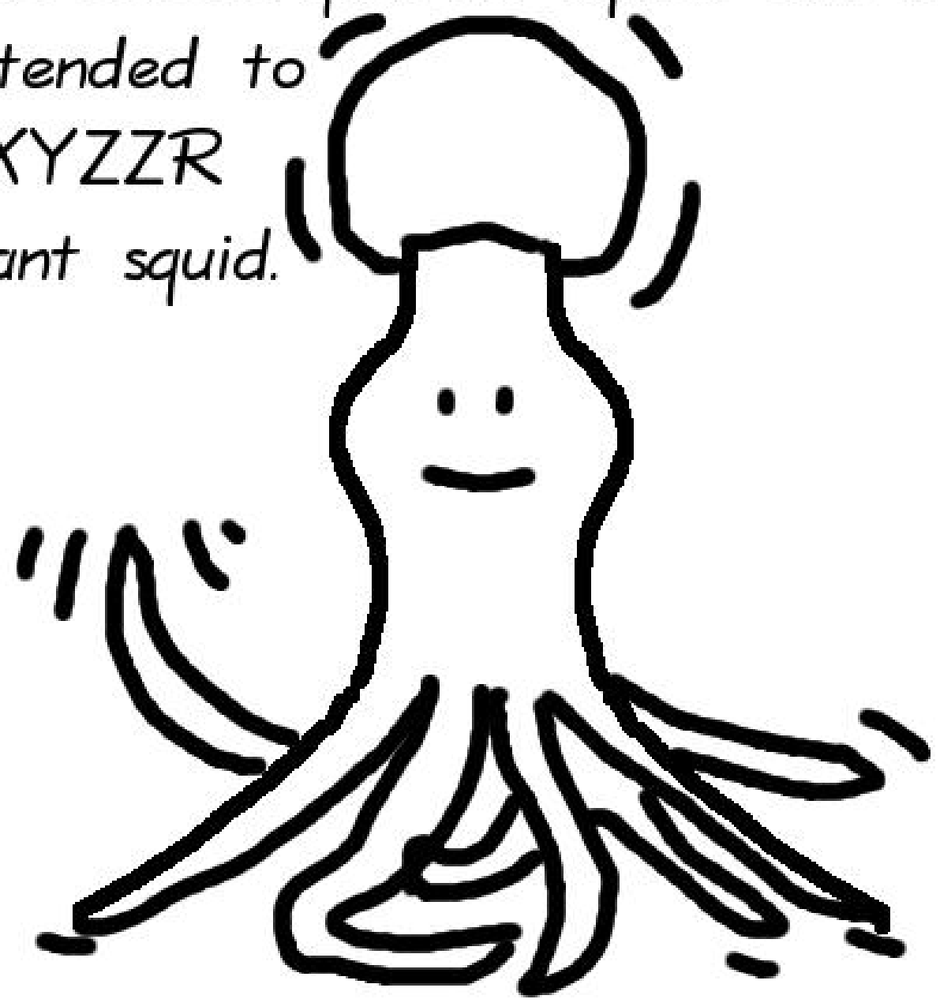
$+ 1$

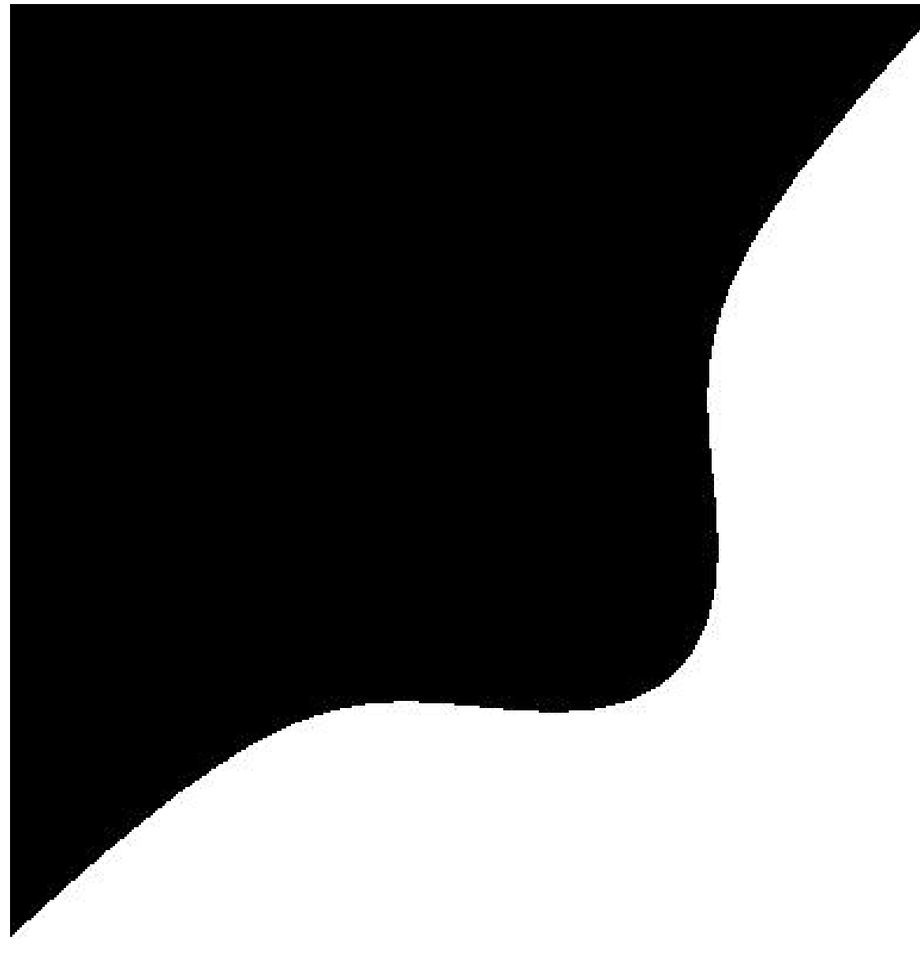
The Jacobi-quartic squid: can be extended to XXYZZR giant squid.



$x^3 - y^3 + 1 = 0.3$

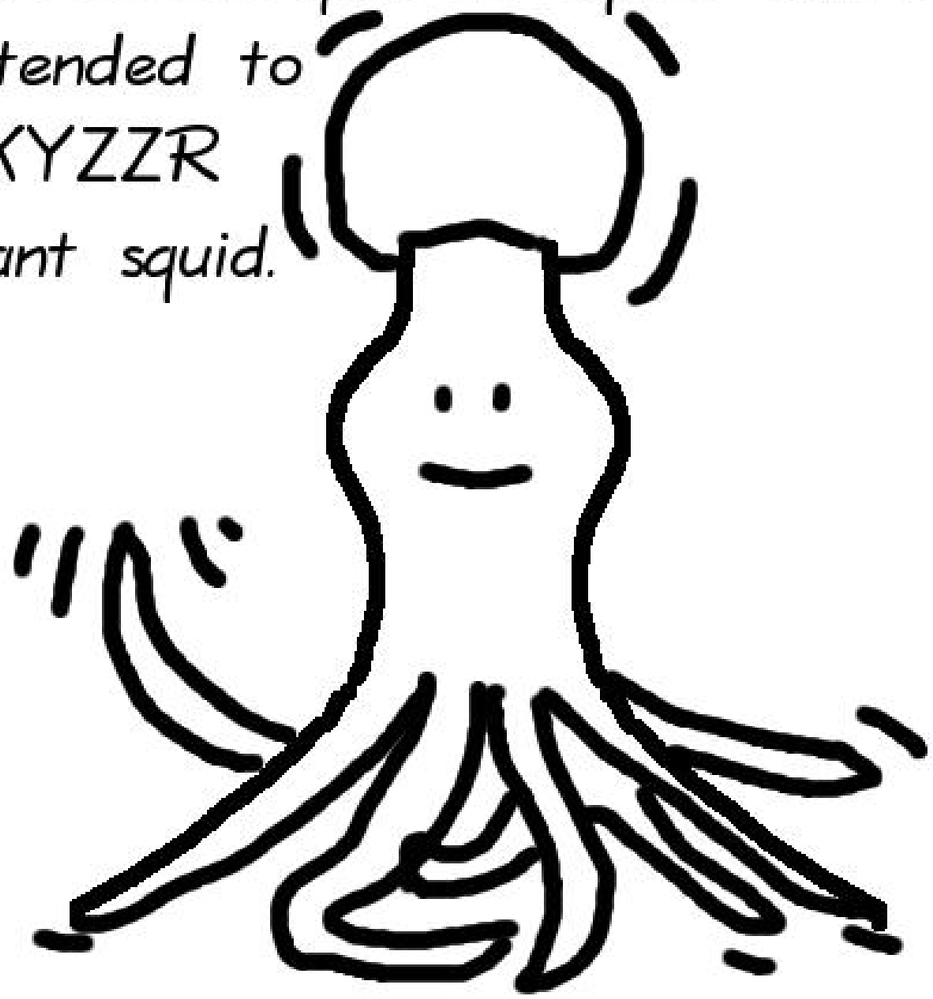
The Jacobi-quartic squid: can be extended to XXYZZR giant squid.

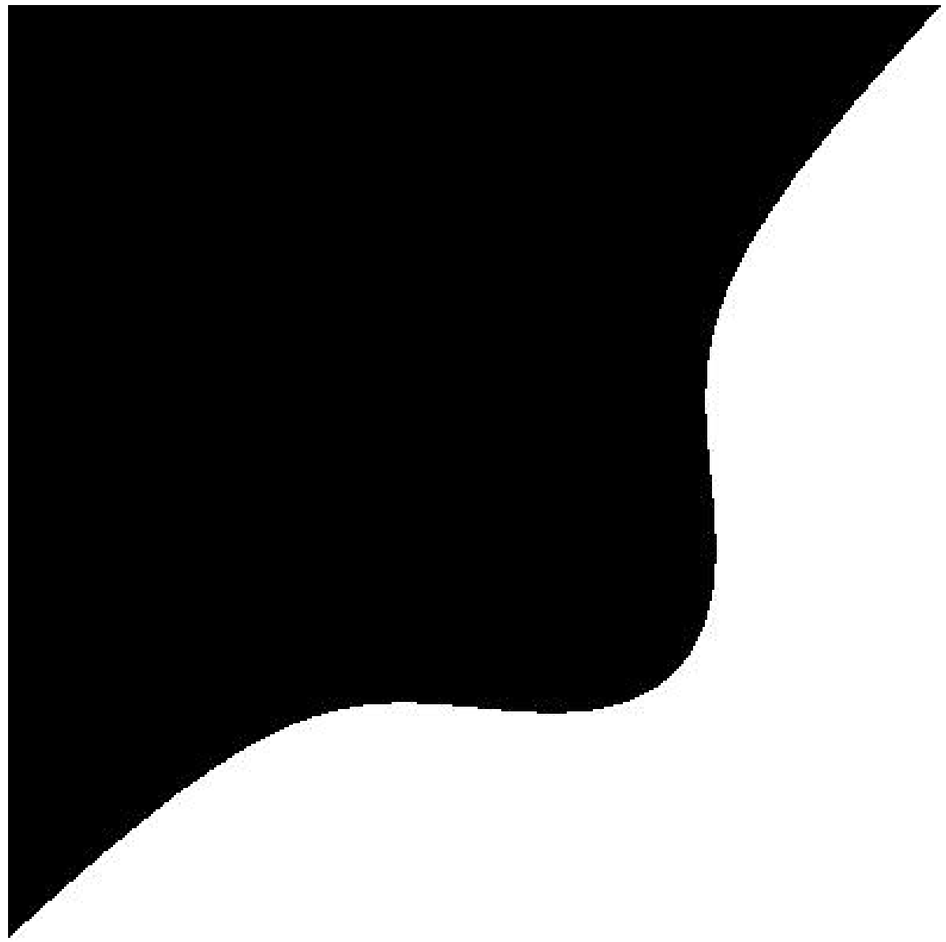




$x^3 - y^3 + 1 = 0.3xy$

The Jacobi-quartic squid: can be extended to XXYZZR giant squid.





$$x^3 - y^3 + 1 = 0.3xy$$

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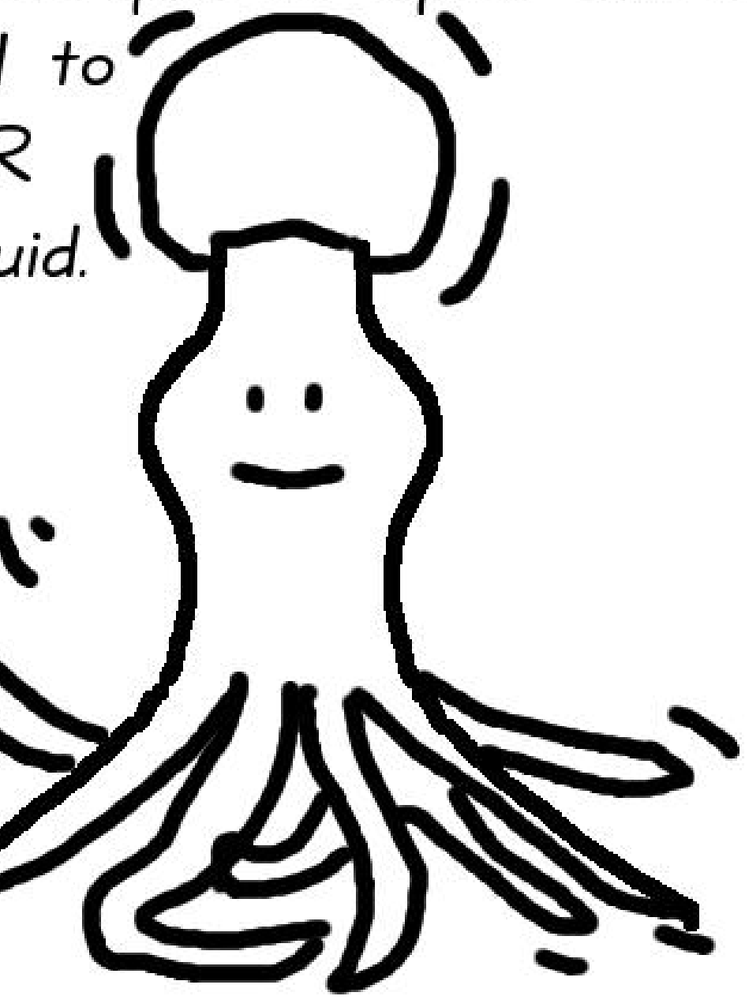
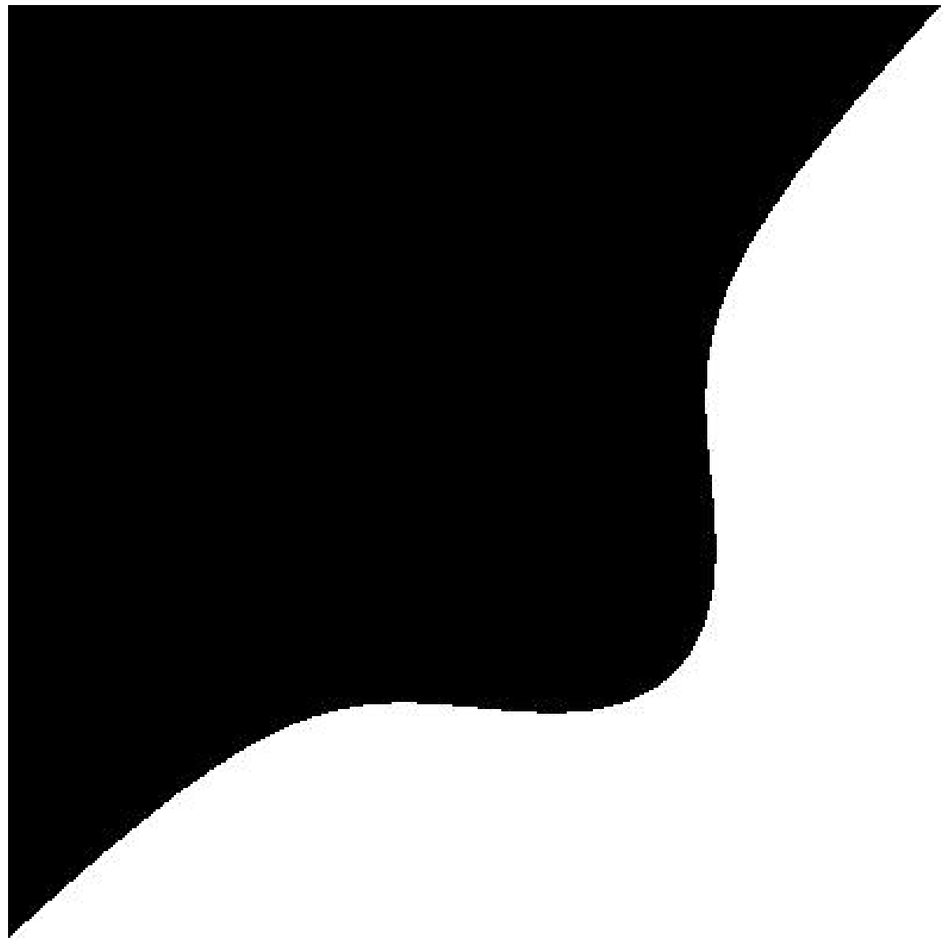
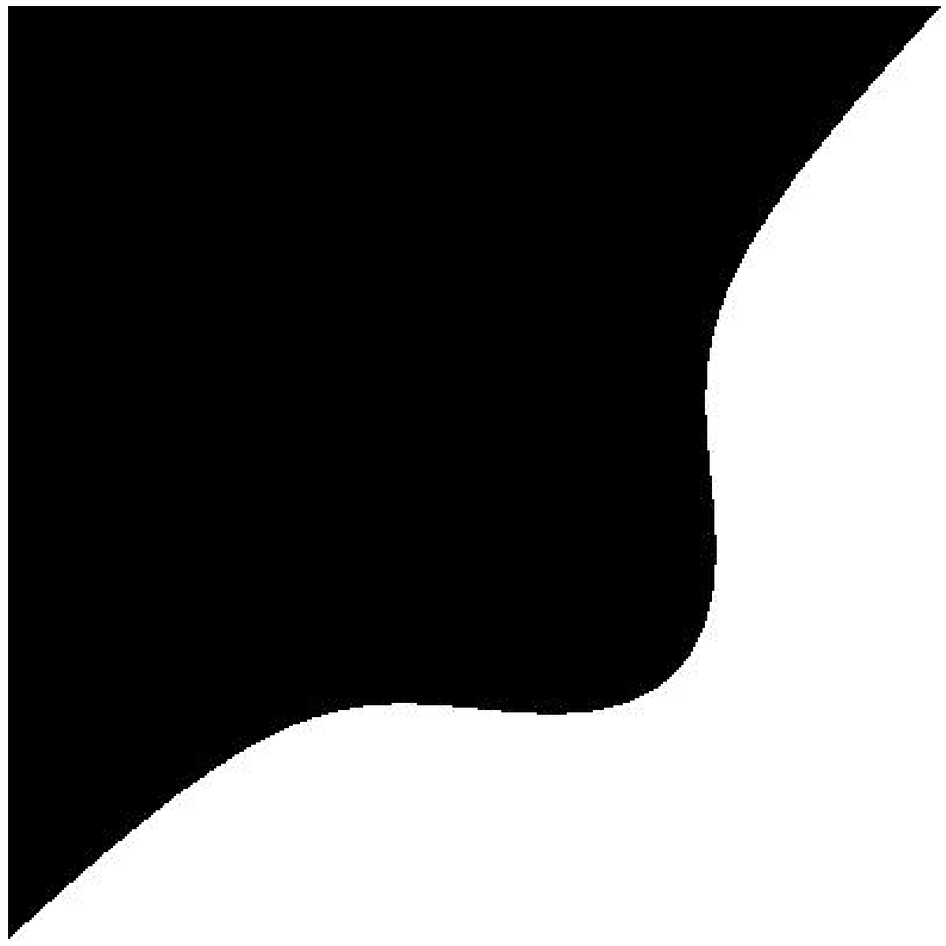




$x^3 - y^3 + 1 = 0.3xy$



$$x^3 - y^3 + 1 = 0.3xy$$

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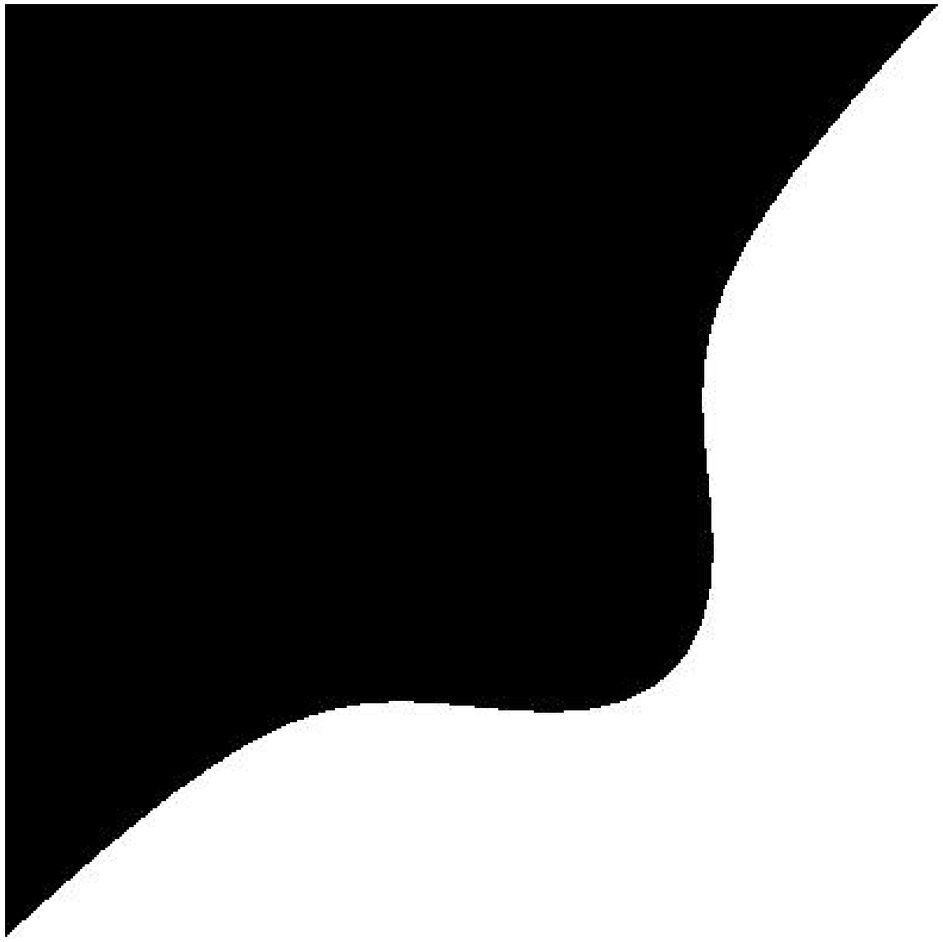

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$x^3 - y^3 + 1 = 0.3xy$

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The Hessian-ray: uniform
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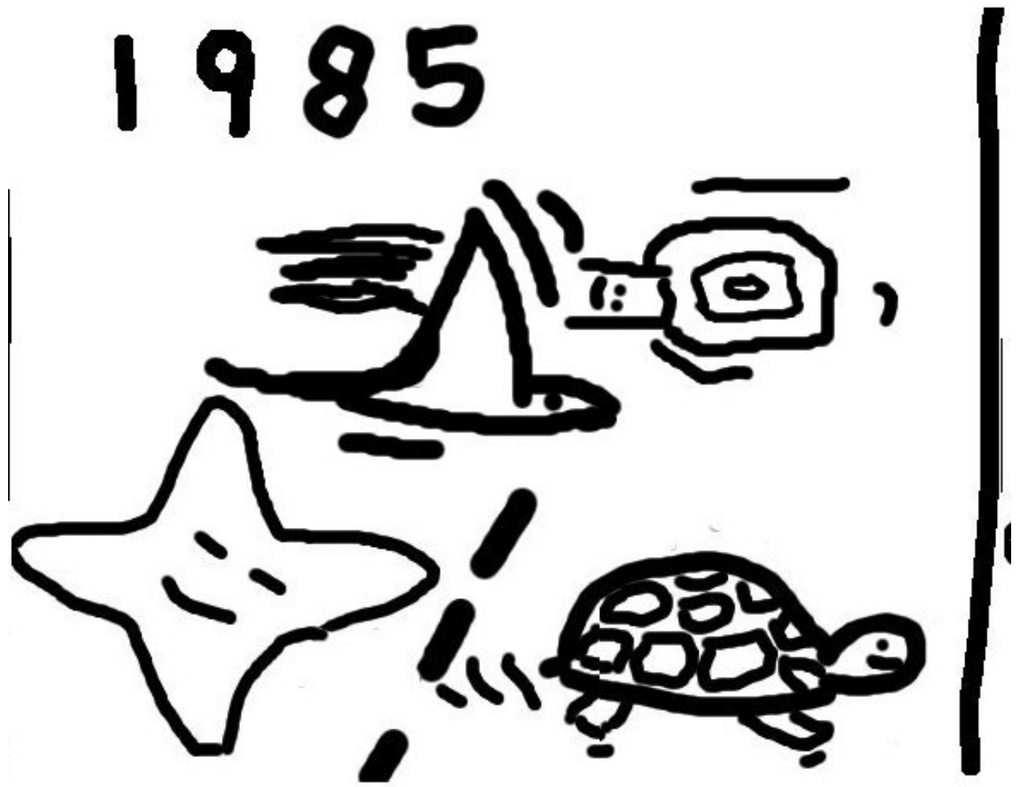
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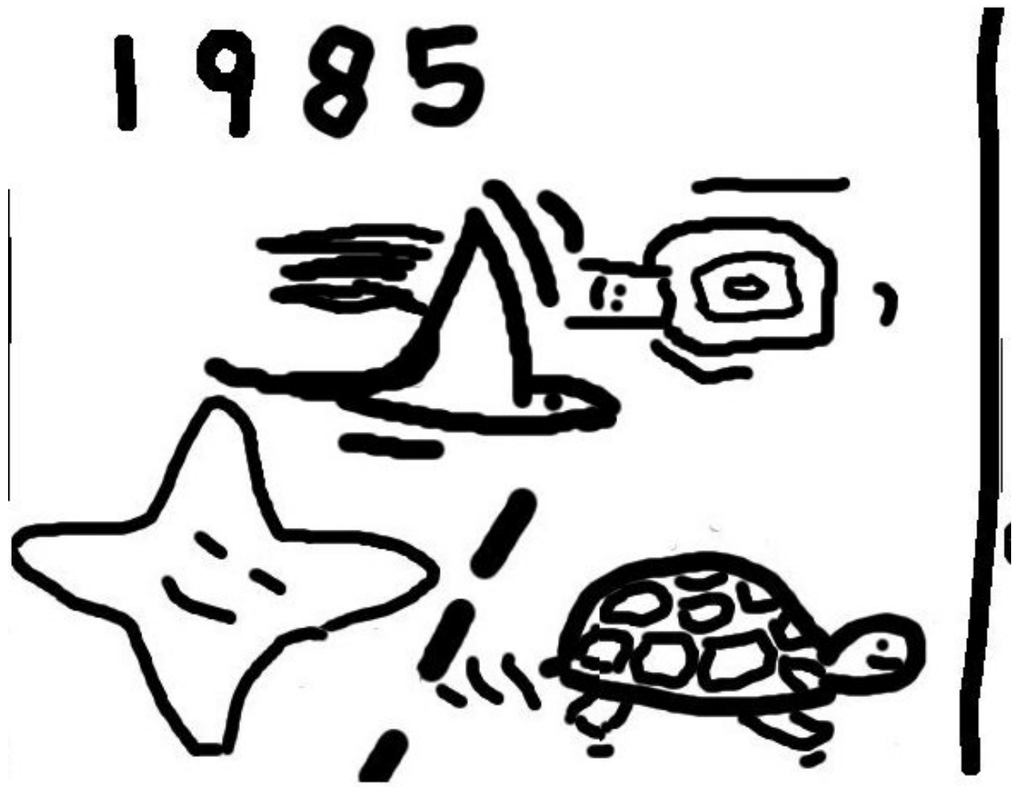
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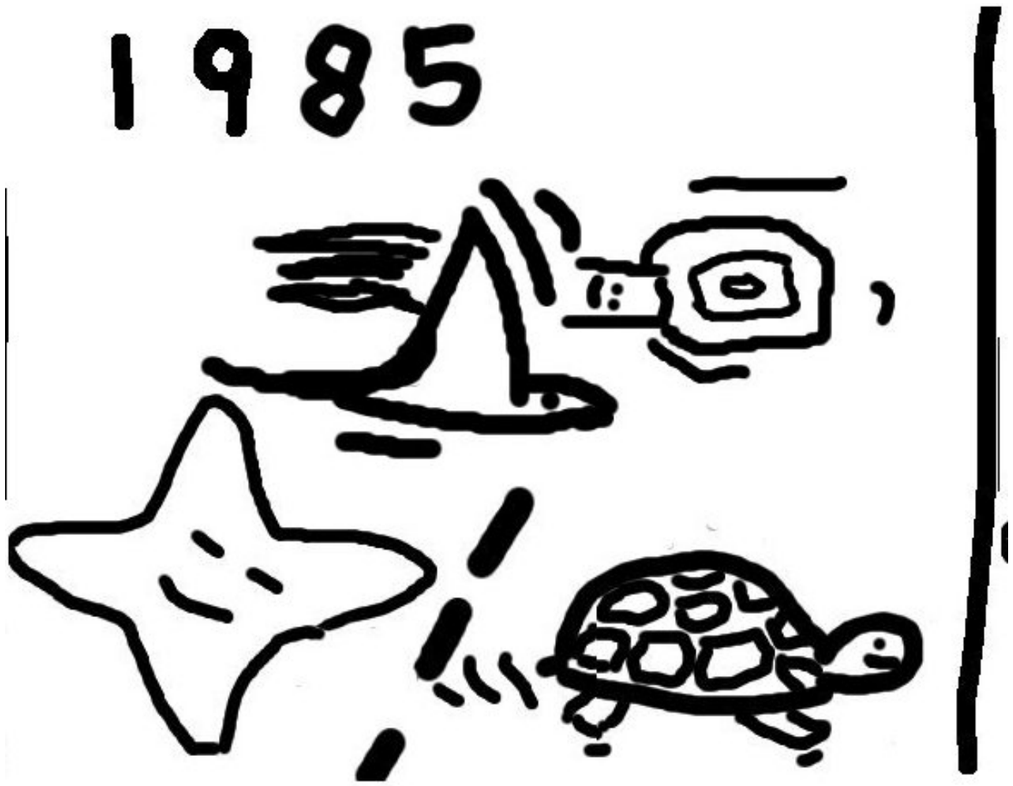
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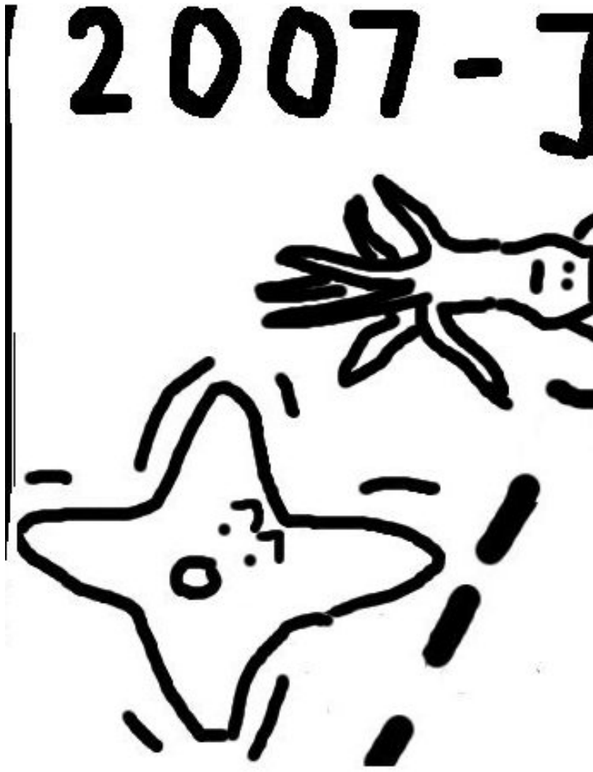
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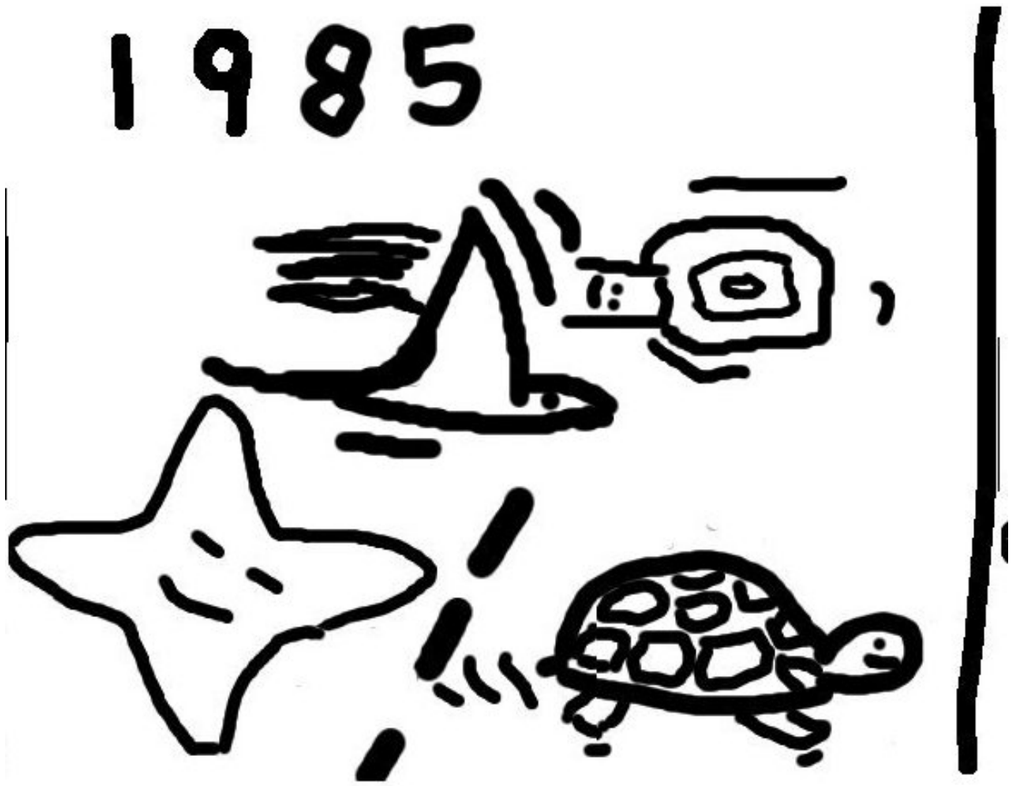
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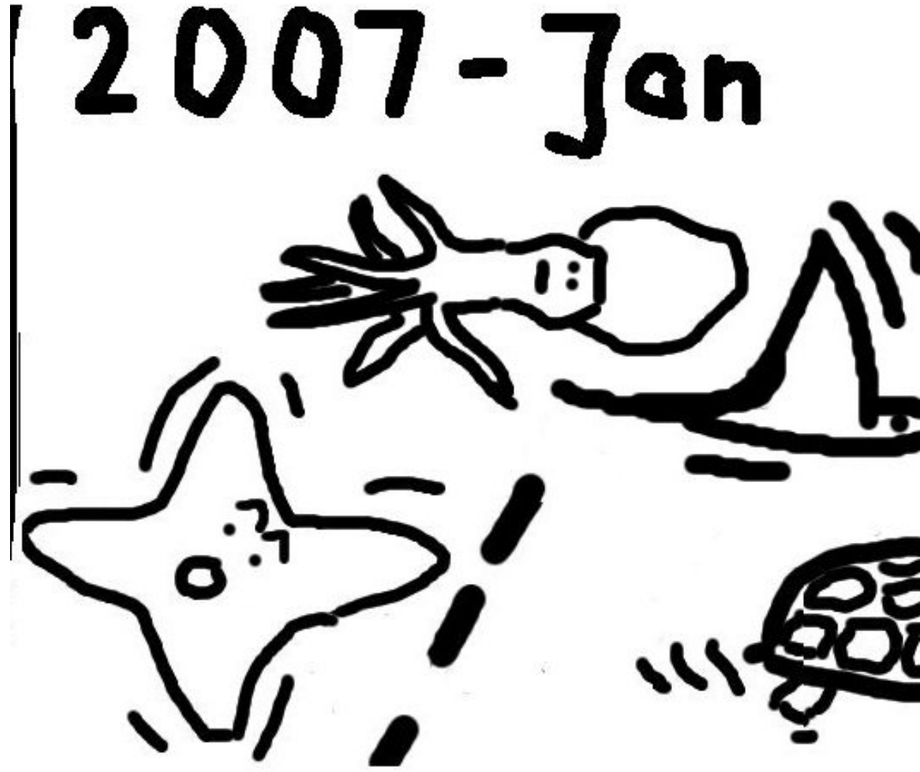
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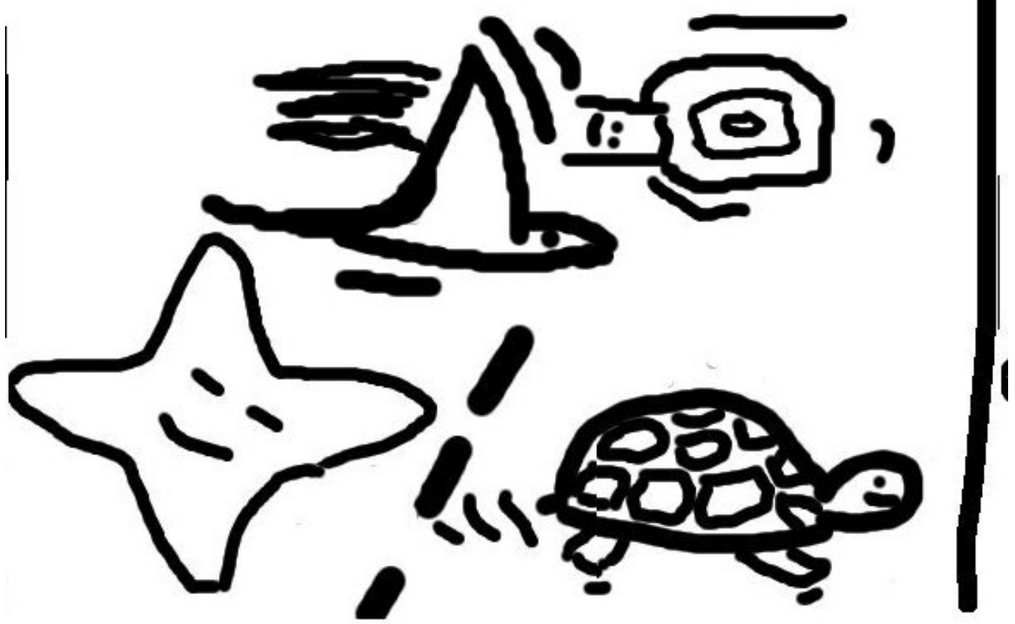

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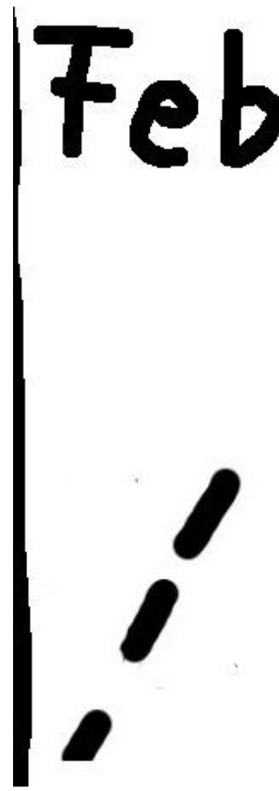
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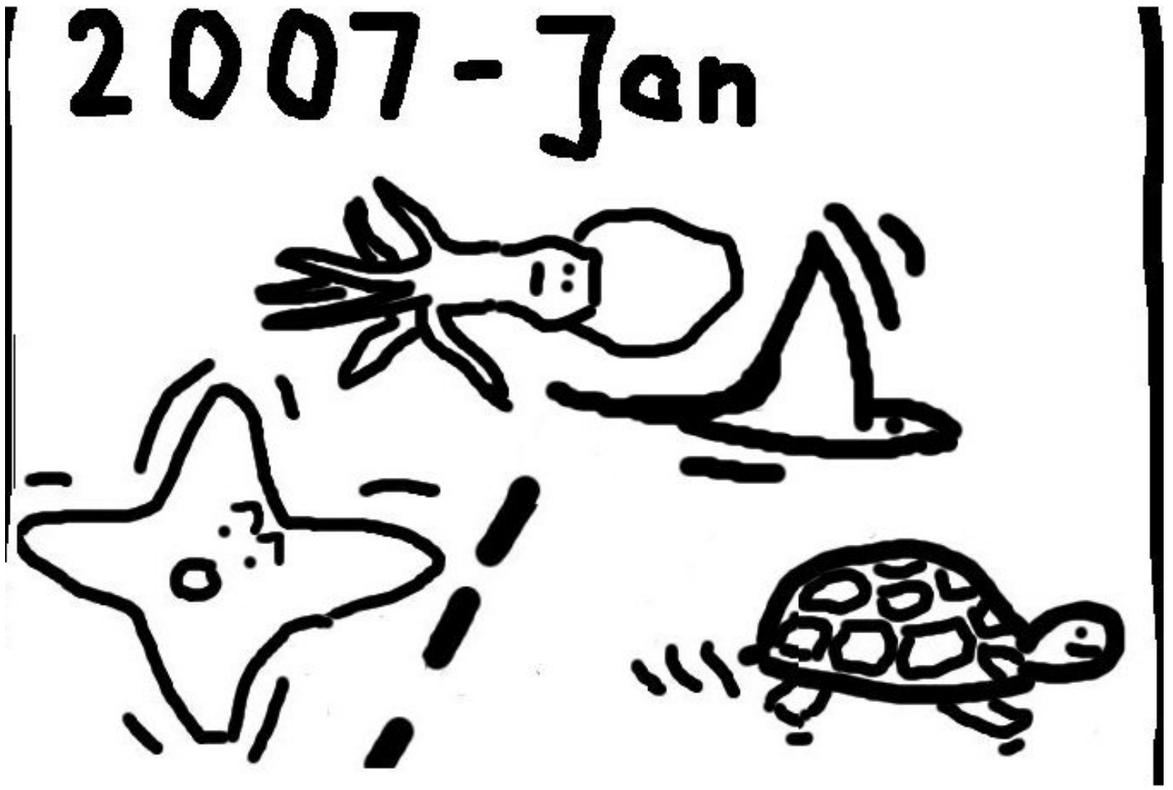
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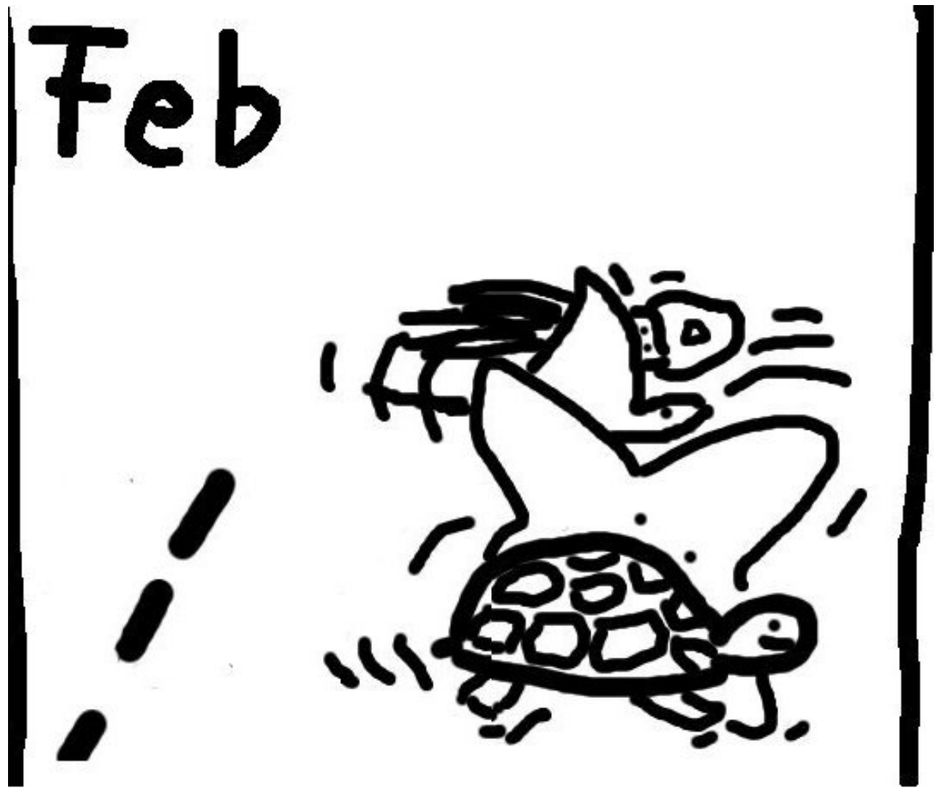
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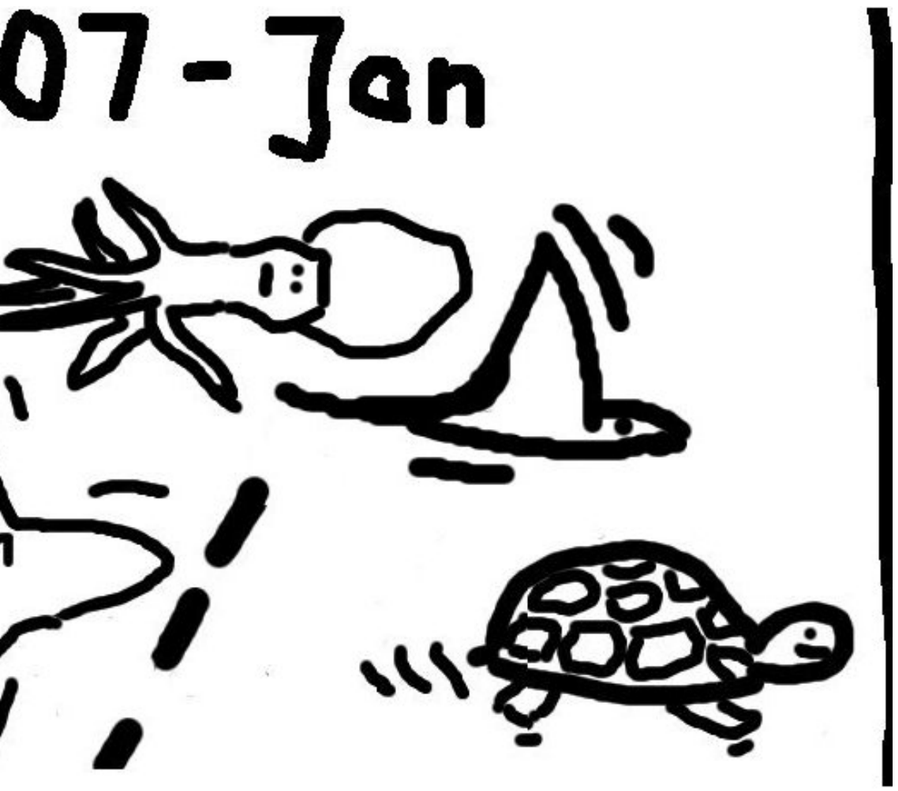
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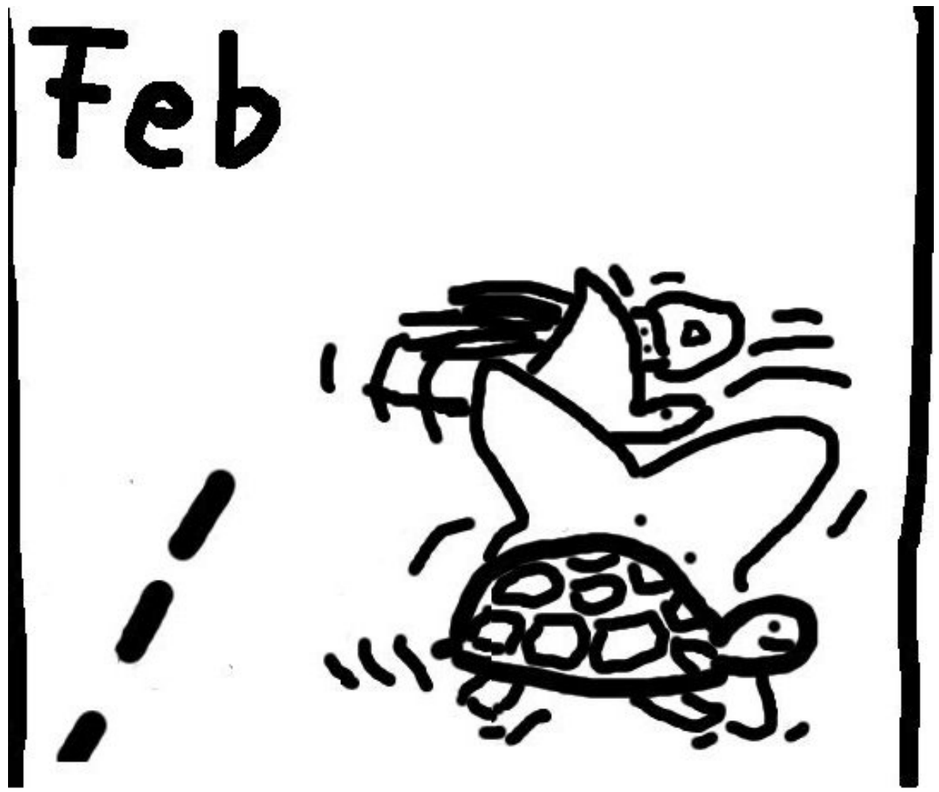
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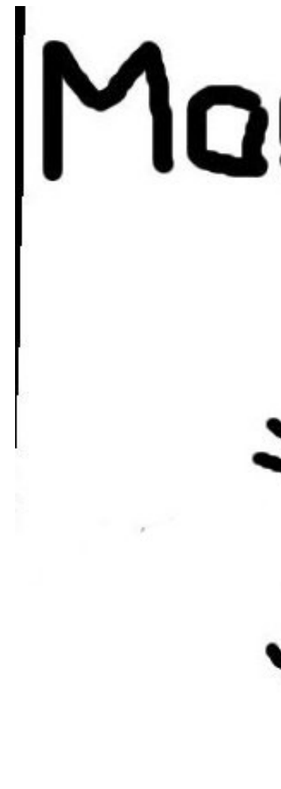
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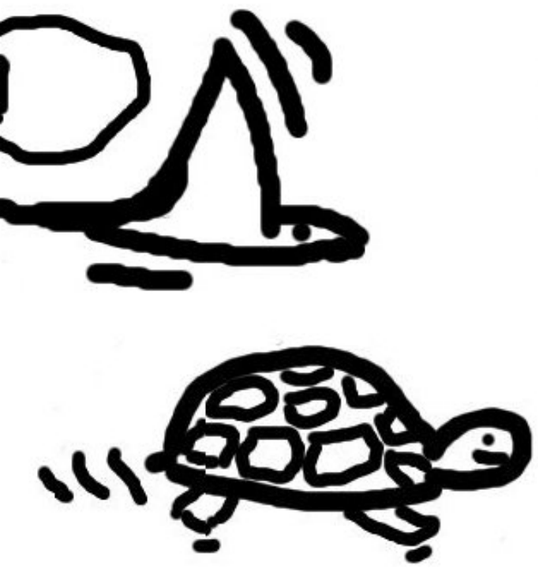

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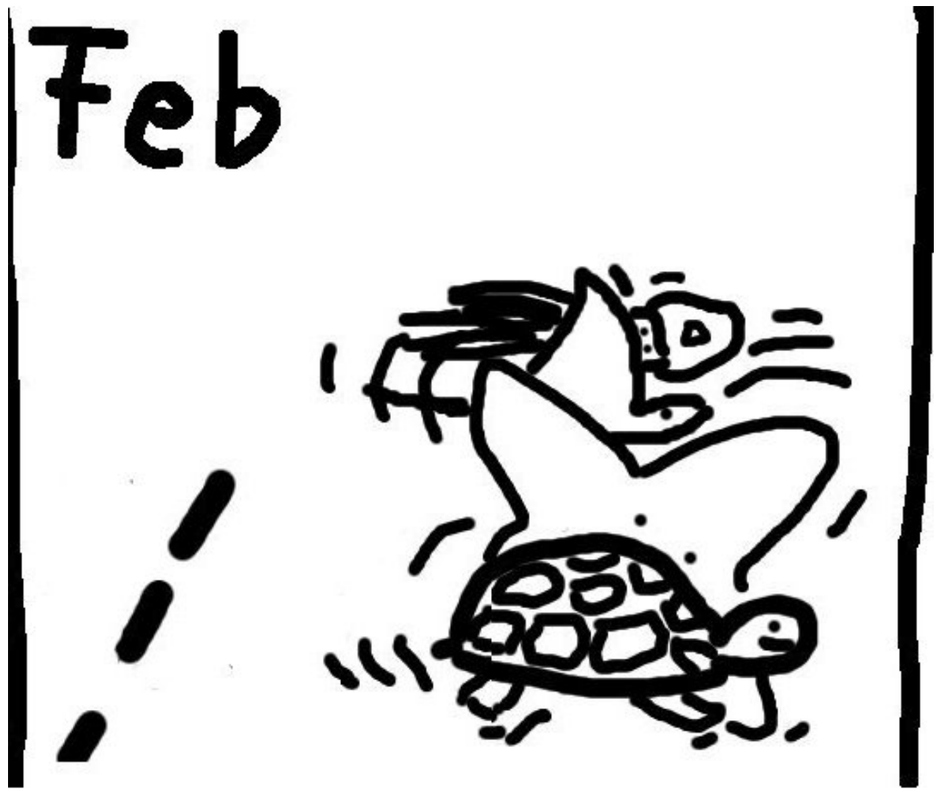
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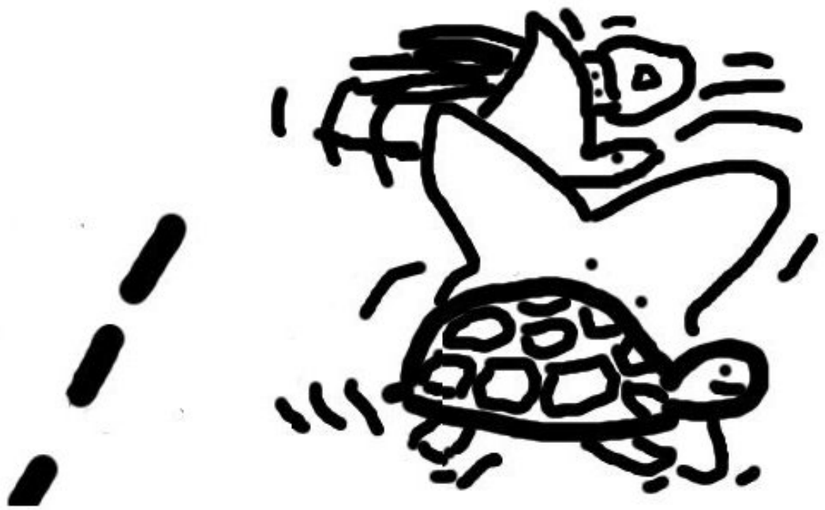

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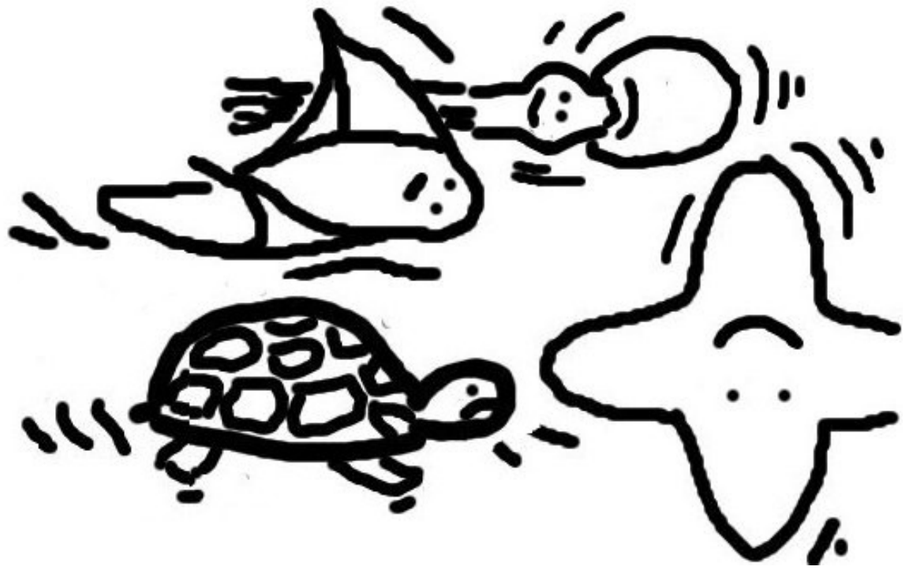

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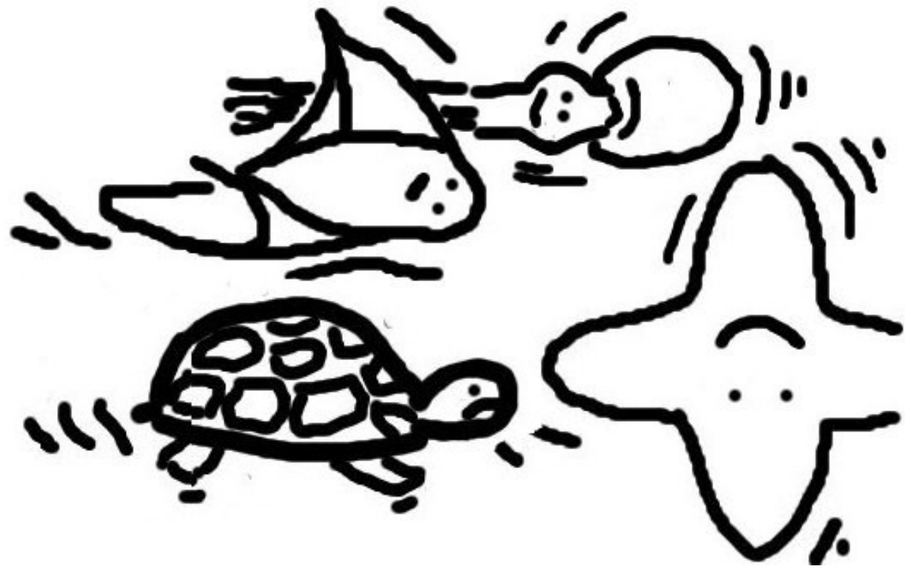

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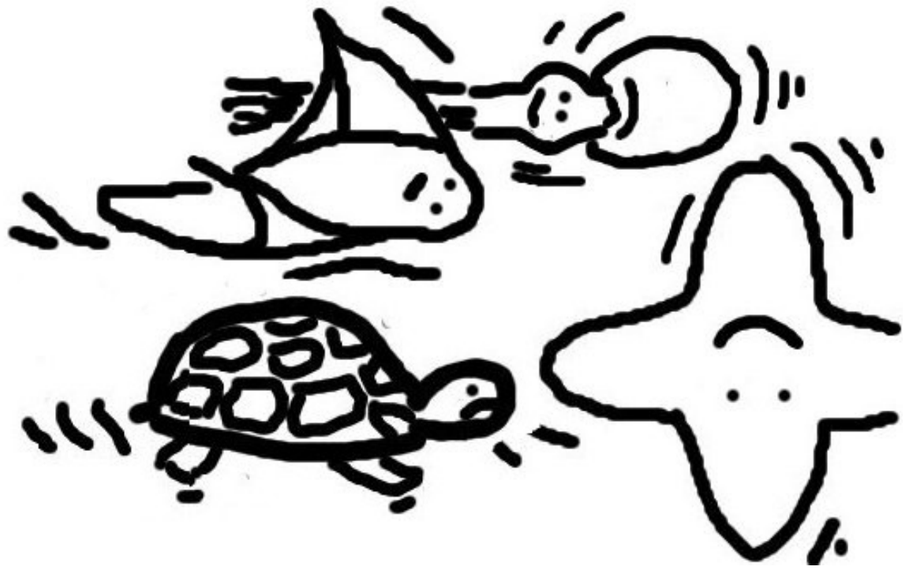

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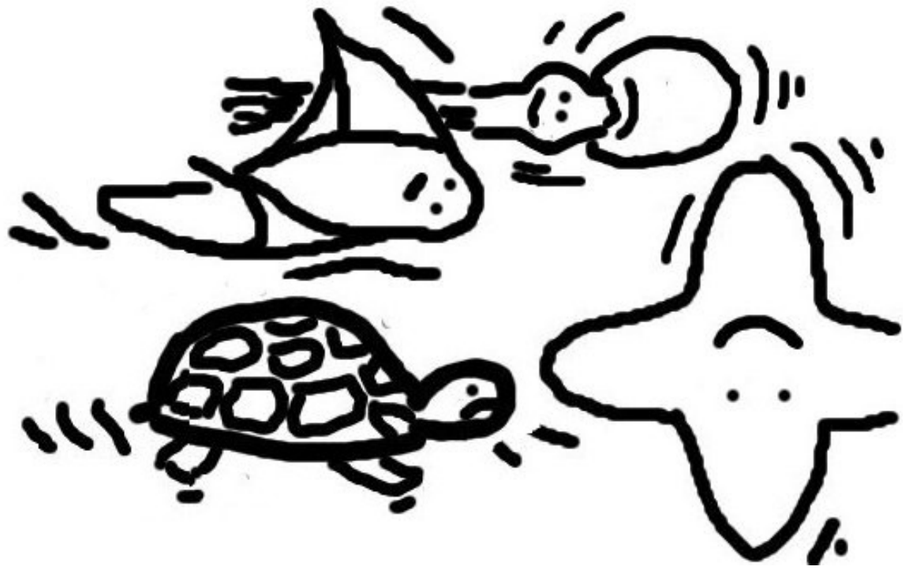

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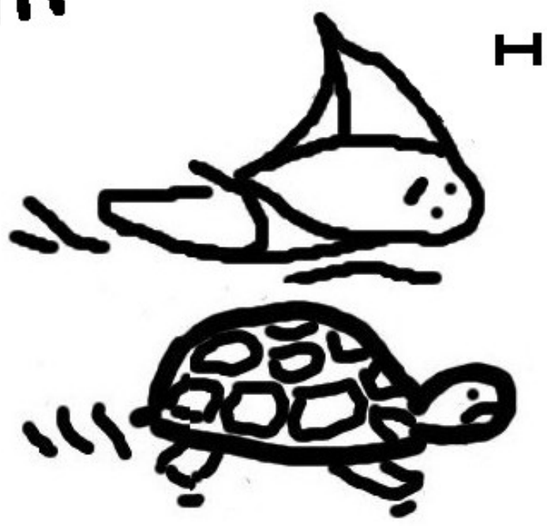

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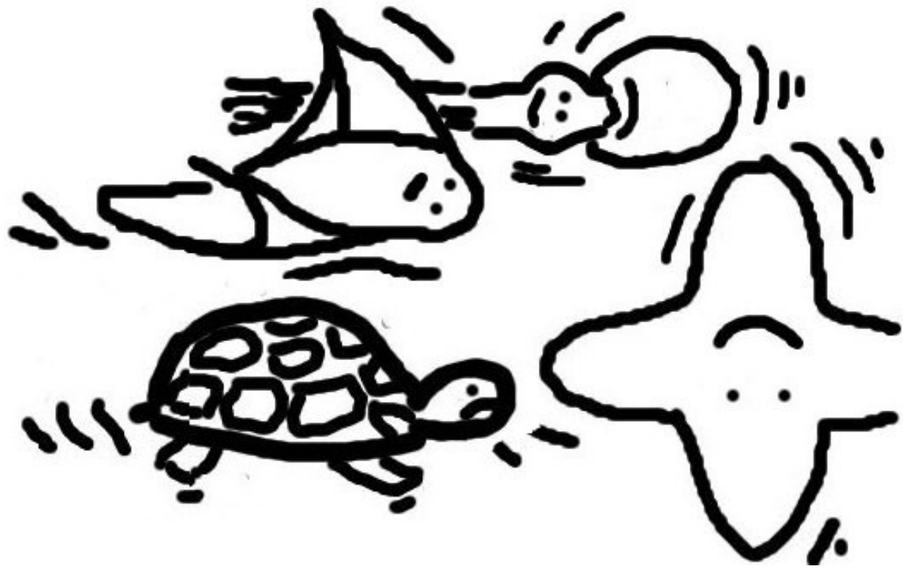

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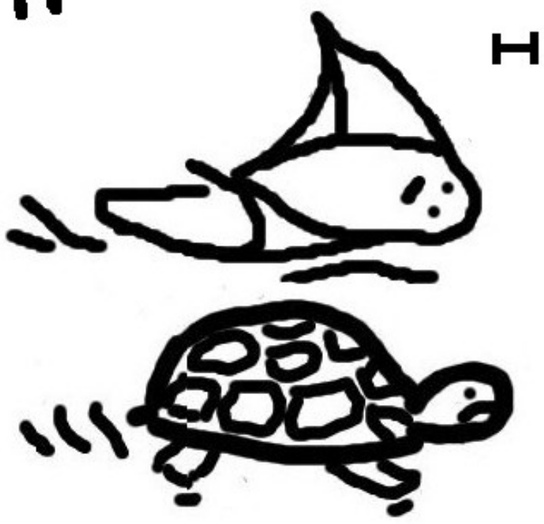

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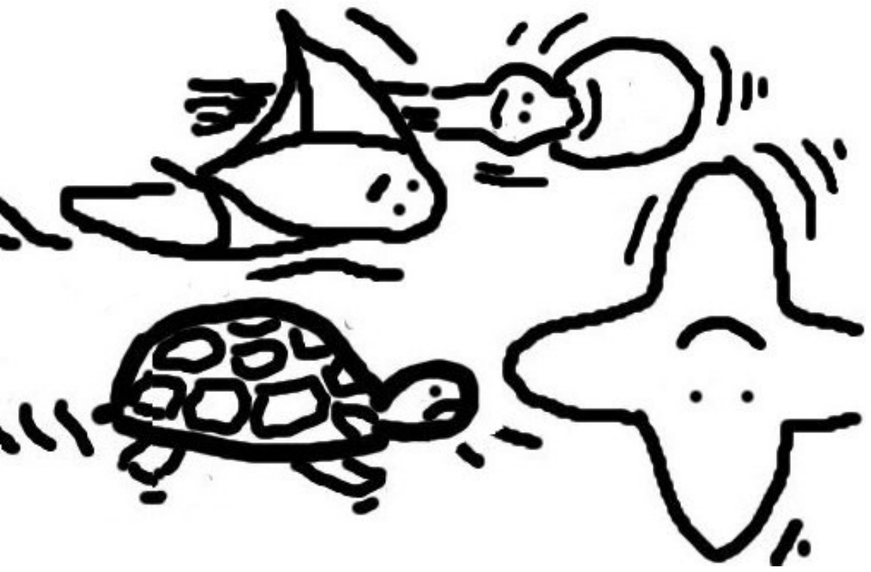

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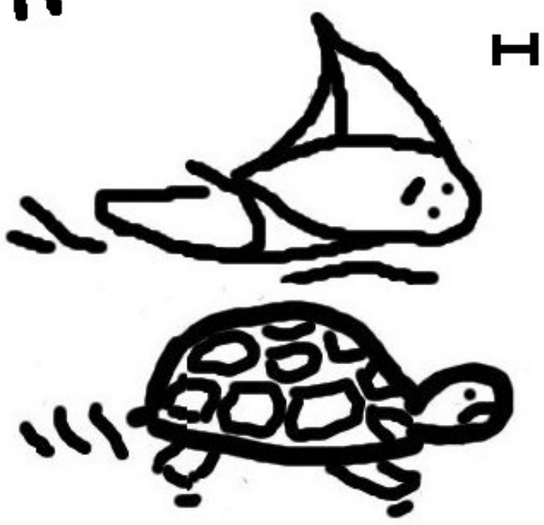

Faster H

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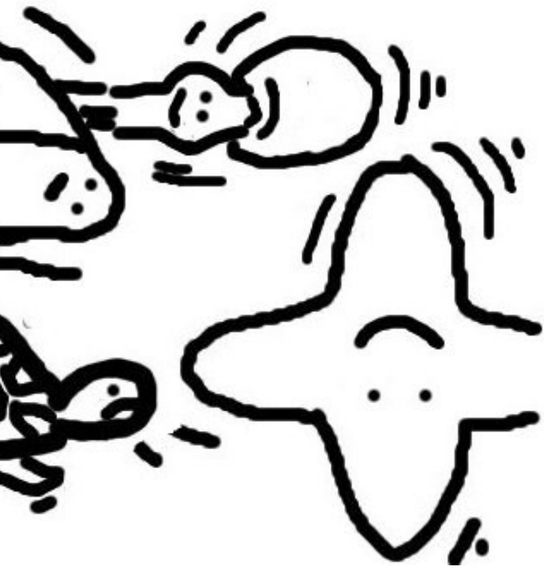

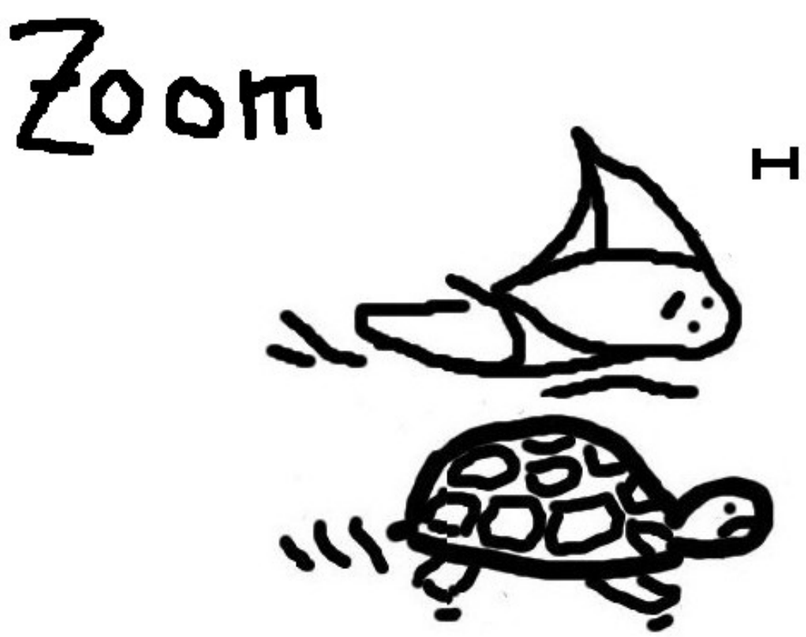

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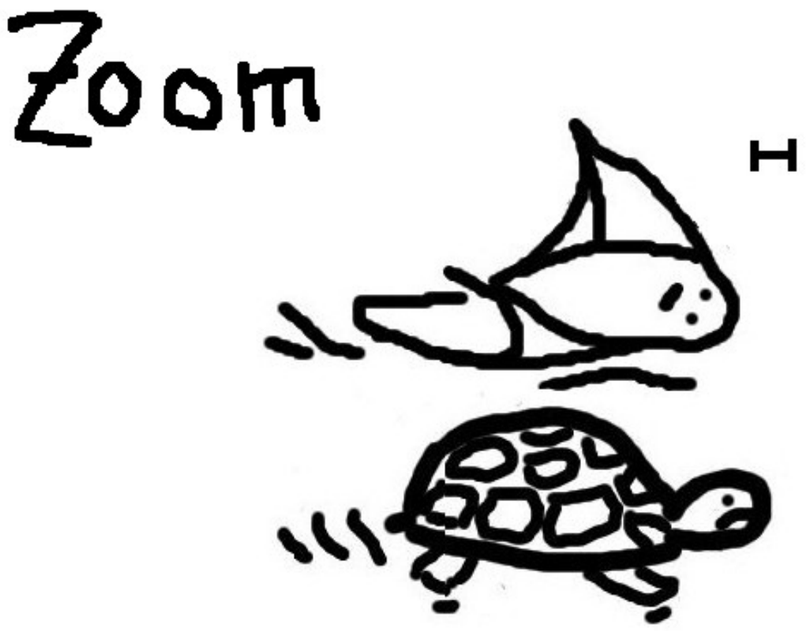
2007 Hisil–Carter–

7.8**M** for DBL.

Faster Hessian arithmetic

2007 Hisil–Carter–Dawson:
7.8**M** for DBL.



Faster Hessian arithmetic

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Faster Hessian arithmetic

2007 Hisil–Carter–Dawson:
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2010 Hisil: 11**M** for ADD.

Faster Hessian arithmetic

2007 Hisil–Carter–Dawson:
7.8**M** for DBL.

2010 Hisil: 11**M** for ADD.

Hessian tied with Weierstrass for DBL-DBL-DBL-DBL-DBL-ADD.

Need to zoom in closer:
analyze exact **S**/**M**, overhead for checking for special cases, extra DBL, extra ADD, etc.

Faster Hessian arithmetic

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Or speed up Hessian more.

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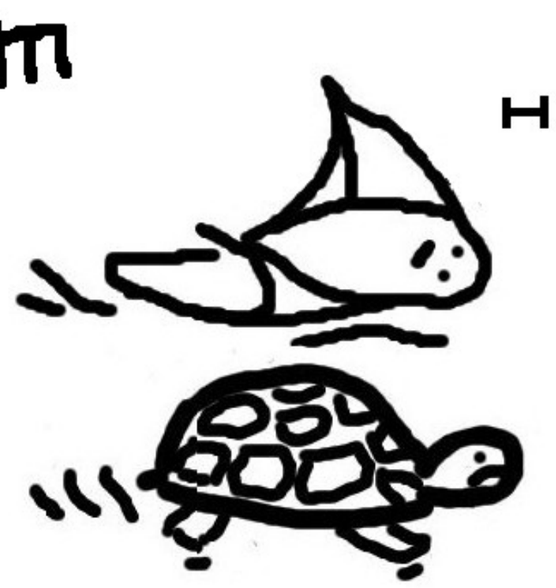
New: 7.6**M** for DBL.



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Faster Hessian arithmetic

2007 Hisil–Carter–Dawson:
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2010 Hisil: 11**M** for ADD.

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New: 7.6**M** for DBL.

New (announced J

Generalize to more
twisted Hessian c
$aX^3 + Y^3 + Z^3 =$
with $a(27a - d^3)$

2007 7.8**M** DBL ic
2010 11**M** ADD g
new 7.6**M** DBL ge

Faster Hessian arithmetic

2007 Hisil–Carter–Dawson:
7.8**M** for DBL.

2010 Hisil: 11**M** for ADD.

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DBL-DBL-DBL-DBL-DBL-ADD.

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Or speed up Hessian more.

New: 7.6**M** for DBL.

New (announced July 2009)

Generalize to more curves:
twisted Hessian curves
$aX^3 + Y^3 + Z^3 = dXYZ$
with $a(27a - d^3) \neq 0$.

2007 7.8**M** DBL idea fails, b
2010 11**M** ADD generalizes,
new 7.6**M** DBL generalizes.

Faster Hessian arithmetic

2007 Hisil–Carter–Dawson:
7.8**M** for DBL.

2010 Hisil: 11**M** for ADD.

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Faster Hessian arithmetic

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DBL-DBL-DBL-DBL-DBL-ADD.

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2007 7.8**M** DBL idea fails, but
2010 11**M** ADD generalizes,
new 7.6**M** DBL generalizes.

Rotate addition law
so that it also works for DBL;
complete if a is not a cube.
Eliminates special-case overhead,
helps stop side-channel attacks.

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3L-DBL-DBL-DBL-ADD.

zoom in closer:
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king for special cases,
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6**M** for DBL.

New (announced July 2009):

Generalize to more curves:

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2007 7.8**M** DBL idea fails, but
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Triplings

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New (announced July 2009):

Generalize to more curves:
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Rotate addition law
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Triplings (assumin

TPL is $P \mapsto 3P$.

2007 Hisil–Carter–
12.8**M** for Hessian

Generalizes to twis

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New (announced July 2009):

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If $VW(V + dU +$
then $aX_3^3 + Y_3^3 +$
where $Q = dU$, R
$S = -(V + Q + R$
$dX_3 = R^3 + S^3 +$
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$(X_3 : Y_3 : Z_3) = 3$

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$aW) = U^3$

$Z_3^3 = dX_3Y_3Z_3$

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2005 Dimitrov–Im
"double-base chai
compute $314159P$
$2^{15}3^2P + 2^{11}3^2P$
$+ 2^4 3^1 P - 2$
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generalized double-base chains:

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9.29**M**/bit for 256-bit scalars.

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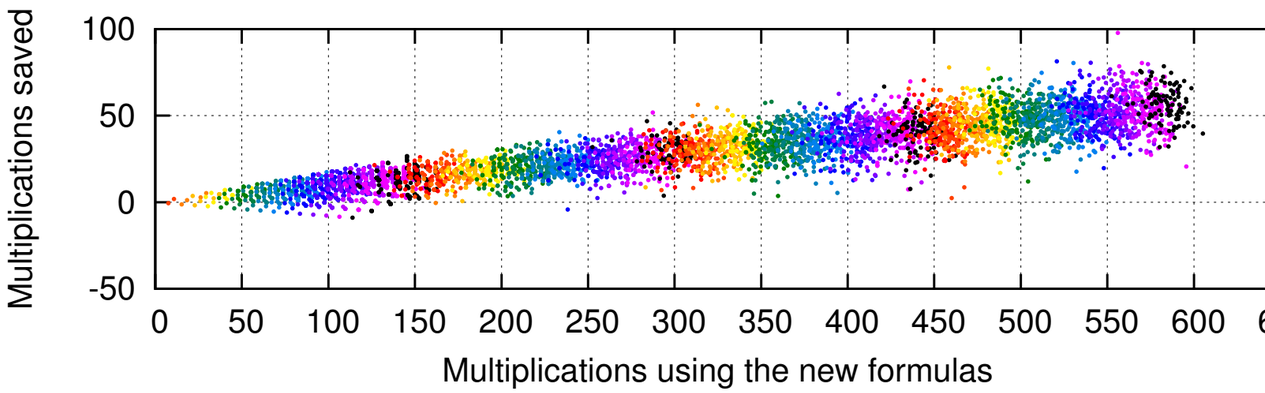

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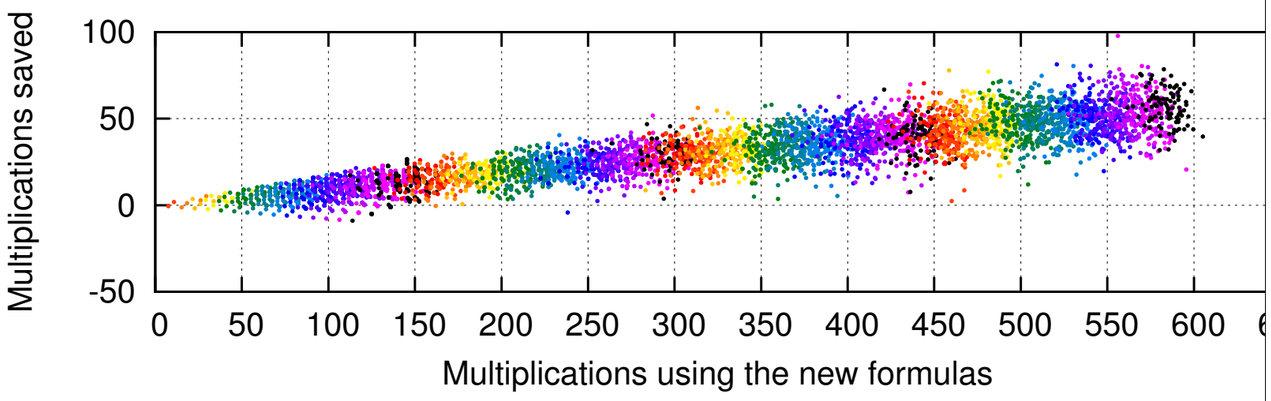

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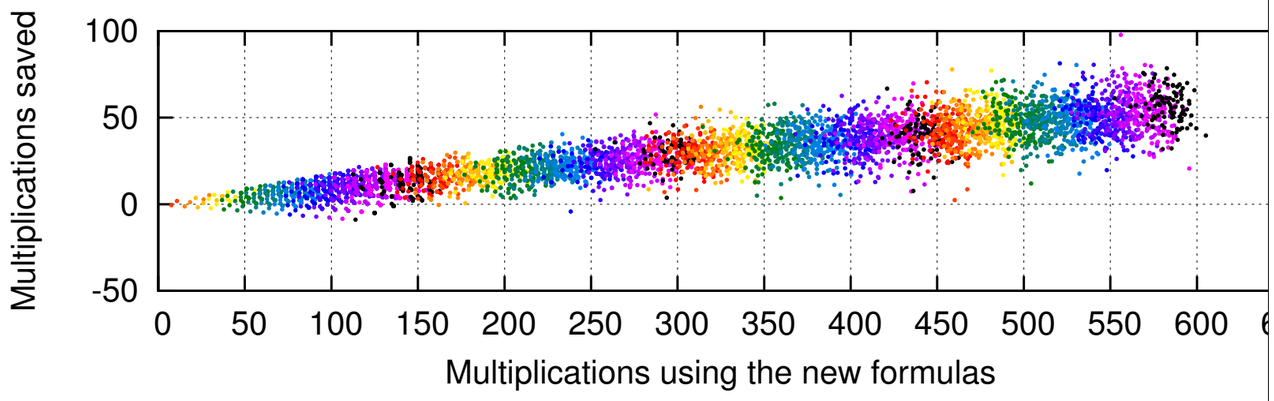

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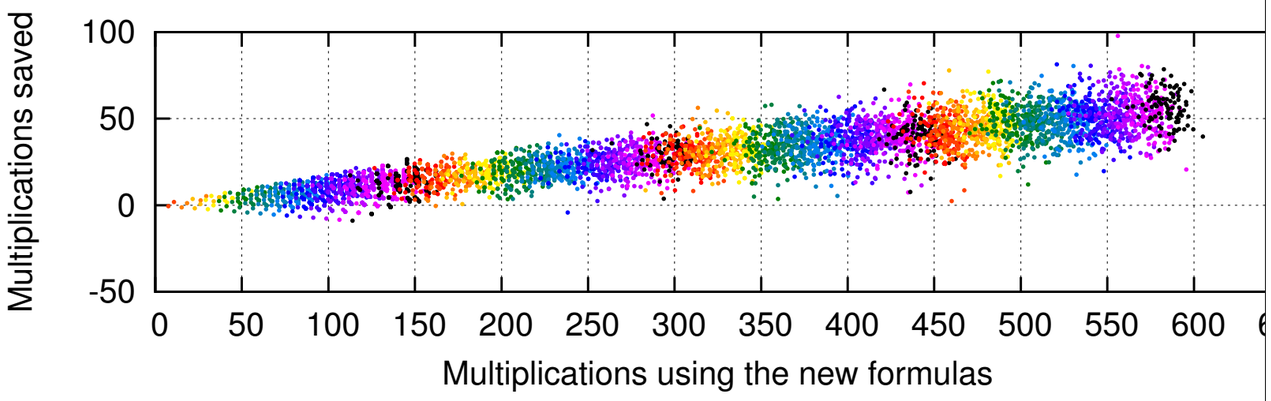

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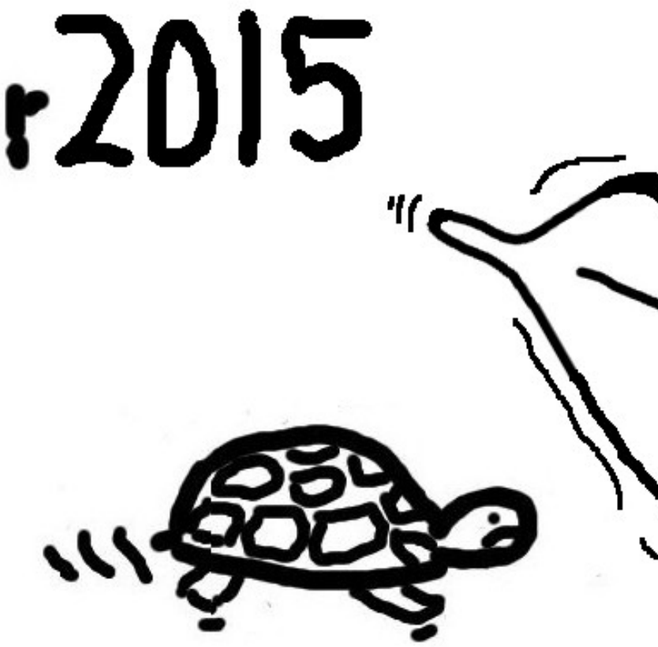

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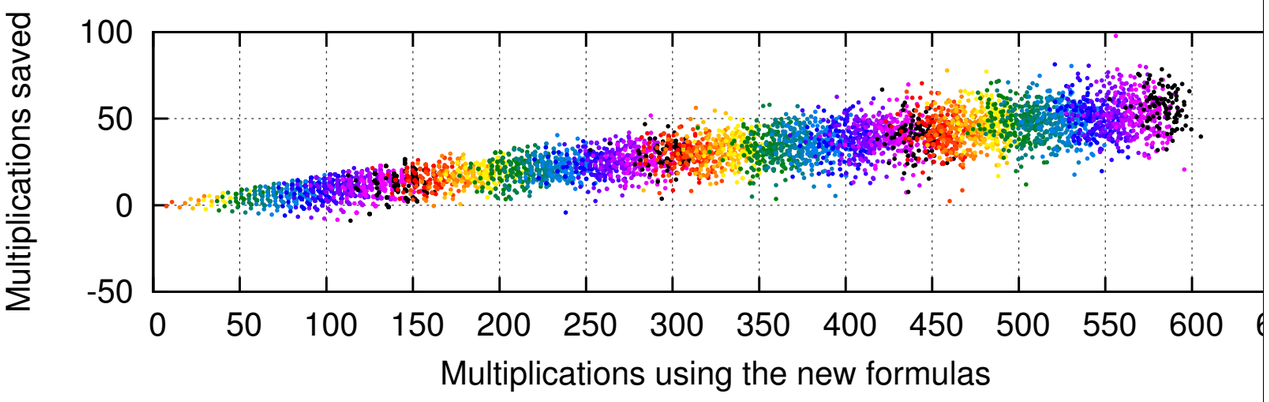

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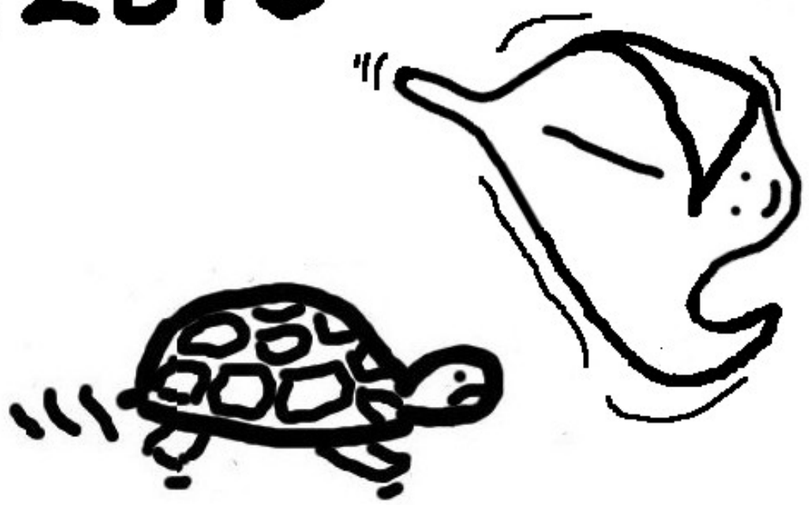

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